RESEARCH STATEMENT
Michael D. Barrus

My main research interests lie in graph theory and combinatorics. A principal theme in my research is the study of structures associated with graphs. Such a structure can be viewed as a collection of parameters or invariants of a graph. Examples include the list of vertex degrees, collections of subgraphs induced in or forbidden for the graph, or the vertex subsets inducing a particular configuration. For such a structure, we examine questions such as (1) for which graphs are the associated structures the same, (2) what properties of the structure imply that two graphs with the same structure must themselves be the same, and (3) given the structure corresponding to some unknown graph, what properties of the graph can we deduce?

Results on these associated objects provide insight into similarities between graphs. The objects are also useful from a computational standpoint, because they may often be stored and manipulated using less memory than needed to store a complete specification of the graph. One can then design algorithms that answer questions about the graph more efficiently than do approaches that require knowing the entire graph.

The structures I have worked with thus far have usually involved some combination of the graph's degree sequence and its induced subgraphs. Properties of one often have implications for the other, though this interplay is not well understood. I study connections between degree sequences and induced subgraphs, and I consider what graph properties are determined when one is given only information about these graph structures. My work can be divided into four areas: graph reconstruction, A_4-structures, forbidden induced subgraphs, and properties of graphic sequences.

Graph reconstruction

One of the most famous open problems in graph theory is the Graph Reconstruction Conjecture [14, 24]. It states that each graph on at least 3 vertices is uniquely determined by its *deck*, the multiset of induced subgraphs (called *cards*) obtained by deleting one vertex from the graph. Results so far have shown how to determine many of a graph's properties from its deck, and the conjecture has been proved for various classes of graphs. However, the general problem remains open at this time.

Initially, there was an analogous reconstruction conjecture for directed graphs. This was disproved in the late 1970s, when Stockmeyer [21] constructed infinite families of digraphs not uniquely determined by their decks. In response, Ramachandran [17] proposed that the reconstruction conjectures be weakened by presenting each vertex-deleted subgraph together with the degree (for graphs) or degree pair (for digraphs) of the deleted vertex in a *degree-associated card*. He conjectured that all graphs and digraphs are uniquely determined by their *degree-associated decks*. For undirected graphs, this conjecture is equivalent to the original statement, since the degree of the missing vertex may be readily computed from the whole deck. For directed graphs, the degree-associated deck gives more information than the deck does. The added degree information is sufficient to reconstruct each of Stockmeyer's digraphs, but whether it suffices for every directed graph is an open question.

One does not always need the entire degree-associated deck to uniquely reconstruct a graph or digraph. In my research, I have examined the *degree-associated reconstruction number* $\mathrm{drn}(G)$ of a graph G, defined in [18] as the minimum number of degree-associated cards that suffice to uniquely determine G. The Reconstruction Conjecture is equivalent to showing that $\mathrm{drn}(G)$ is defined for each graph G. In [8] my advisor Douglas B. West and I observed that $\mathrm{drn}(G) \leq 2$ for almost all graphs G (asymptotically), and we characterized the graphs for which $\mathrm{drn}(G) = 1$. We obtained $\mathrm{drn}(G)$ for all G in various graph classes. In particular, it is known that $\mathrm{drn}(T) \leq 3$ for any tree T; we showed that for all caterpillars but one, $\mathrm{drn}(T) \leq 2$. We conjecture that there are relatively few trees with $\mathrm{drn}(T) = 3$, and we are investigating whether these may be easily characterized. We also showed that $\mathrm{drn}(G) \geq 3$ for all vertex-transitive graphs other than complete graphs, and that equality holds for hypercubes. In a sense, this says that vertex-transitive graphs are generally harder to reconstruct. In the future I plan to extend our results on vertex-transitive graphs. Understanding better the difficulties in reconstructing these graphs may lead to better understanding of how to reconstruct any graph.

A_4-structures of graphs

The *P_4-structure* of a graph G is a 4-uniform hypergraph having the same vertex set as G in which four vertices form an edge if and only if they induce a path (a P_4) in G. In work related to Berge's Strong Perfect Graph Conjecture, Chvátal [11] introduced the P_4-structure and conjectured that two graphs having the same P_4-structure are either both perfect or both imperfect (this was later proved by Reed [20]). Research on the P_4-structure has since grown beyond a focus on perfect graphs; the P_4-structure has been used to define several graph classes with interesting structural properties in which several optimization problems can be solved more efficiently than in general. It has also appeared in several schemes of graph decomposition (partitioning the vertex set of a graph into subsets with prescribed properties).

An induced P_4 in a graph gives rise to an *alternating 4-cycle*, a configuration on four vertices in which two edges and two non-edges alternate in a cyclic fashion. In [7], we defined the *A_4-structure* of a graph G by modifying the definition of the P_4-structure to include as edges the vertex sets of all alternating 4-cycles. We showed that the A_4-structure has strong ties to the *canonical decomposition* of a graph, as defined by Tyshkevich [22, 23]; in particular, a graph is canonically indecomposable if and only if its A_4-structure is connected. Furthermore, a graph has the same A_4-structure as a split graph if and only if the "core" of its canonical decomposition is split or is a disjoint union of stars or the complement of such a graph.

At present, I am working on characterizing the *A_4-balanced* and the *P_4-balanced* graphs, those whose vertex set can be partitioned into two subsets so that each alternating 4-cycle or P_4, respectively, has two vertices in each subset. The class of P_4-balanced graphs contains both the split-perfect (Brandstädt and Le [9]) and bipartite-perfect (Le [15]) graphs, and both the A_4-balanced and P_4-balanced graphs have interesting structural and algorithmic properties. In the future, I also hope to characterize all hypergraphs that are A_4-structures for graphs and provide techniques for obtaining all realizations of a given A_4-structure.

Degree sequences and minimal forbidden subgraphs

One trend in the development of graph theory has been to study classes of graphs with widespread interest and to give necessary and sufficient conditions for membership in them. The degree sequence of a graph may be particularly useful for these purposes, as conditions depending only on the degree sequence are easy to state and can often be tested by linear-time algorithms. Call a graph family $\mathcal{G}$ *degree-determined* if the question of whether a graph H belongs to $\mathcal{G}$ can be answered knowing only the degree sequence of H. Unfortunately, most graph classes of broad interest are not degree-determined. Some, however, are: examples include the complete, split, matrogenic, and matroidal graphs. Each of these families has a linear-time recognition algorithm based on a degree sequence characterization.

The graph families listed above are all hereditary (meaning they are closed under taking induced subgraphs). For every hereditary class $\mathcal{G}$ there is a minimal set $\mathcal{F}$ of graphs such that a graph H is in $\mathcal{G}$ if and only if it is $\mathcal{F}$*-free*, i.e., it contains no induced subgraph isomorphic to an element of $\mathcal{F}$. We call the elements of $\mathcal{F}$ *obstructions* for the class $\mathcal{G}$.

As we have observed, several hereditary families are degree-determined, though not all are. Together with Mohit Kumbhat and Stephen Hartke, I have sought to characterize degree-determined hereditary families by studying their associated sets of obstructions. In [4] we defined a set $\mathcal{F}$ of graphs to be *degree-sequence-forcing (DSF)* if the class of $\mathcal{F}$-free graphs is degree-determined. We showed that every DSF set must contain (1) a disjoint union of complete graphs, (2) a complete multipartite graph, (3) a forest of stars, and (4) the complement of a forest of stars. We characterized the DSF sets of size at most two and the *non-minimal* DSF triples, i.e., DSF sets of three graphs with a proper DSF subset [4, 6]. In addition, we showed [5] that for any natural number k, there are finitely many minimal DSF k-sets. Currently, we are working on identifying all minimal DSF triples, with an ultimate goal of providing a simple characterization for all DSF sets.

Our results on DSF sets have been largely existential in nature; in most cases we prove that a set $\mathcal{F}$ is DSF without obtaining a degree sequence characterization for the $\mathcal{F}$-free graphs. One exception to this is my work on two particular DSF triples [2] that, when forbidden, generate graph families that contain the threshold graphs and are related to matrogenic graphs. I provided structural and degree sequence characterizations for these families that generalize well-known results on threshold graphs.

For certain hereditary families $\mathcal{G}$, the degree sequence of a graph H determines not only whether H is in $\mathcal{G}$, but also how many edges must be added to or deleted from H to produce a graph in $\mathcal{G}$. When $\mathcal{G}$ is the class of split graphs, this parameter is known as the *splittance* of H; Hammer and Simeone [13] defined the splittance and gave a formula for it in terms of the degree sequence. Hereditary families (like the split graphs) having this additional degree sequence property have sets of obstructions that are necessarily DSF; in the future, I hope to characterize these special DSF sets.

Properties of degree sequences

A list of positive integers is *graphic* if it is the degree sequence of some (simple) graph. Not every list is graphic; there are many equivalent sets of conditions for deciding this for a given list (for a summary, see [16]). In particular, short sequences are often not graphic. Using a method of Aigner and Triesch [1], my coauthors and I showed [3] that a partial converse holds: lists with even sum that are long enough (in terms of the largest and smallest entries) are graphic. In general, the necessary length for realizability of a list is quadratic in the largest entry; this relationship is linear when the list contains all values between the largest and smallest entries. Our bounds are best possible in terms of these parameters. We also obtained partial results towards an improvement of these bounds when the maximum difference between consecutive terms in the list is prescribed, including solving the problem completely when the maximum difference is 1.

Future work

I expect that the interplay between degree sequences and induced subgraphs will continue to be an active area of research for some time. There is still more to be done for each of the topics I have touched on above. Furthermore, while many results are known that ensure that *one* graph having a given degree sequence satisfies a given property, considerably fewer are known that characterize degree sequences for which *all* realizations have the property (see, for example, the survey in [19]). Fewer still are those results of this latter type when the property is the *presence*, rather than absence, of a particular induced subgraph or subgraphs. Results in this direction could lead to better bounds in terms of the degree sequence on such graph parameters as the matching, clique, and independence numbers, improving upon results such as those of Caro and Wei [10, 25] and Favaron et al. [12].

References

[1] M. Aigner and E. Triesch, Realizability and uniqueness in graphs, Discrete Math. 136 (1994), no. 1-3, 3–20.

[2] M. D. Barrus, Chair- and kite-free pseudosplit graphs, submitted.

[3] M. D. Barrus, S. G. Hartke, K. Jao, and D. B. West, Length thresholds for graphic lists with fixed maximum and minimum entries, in preparation.

[4] M. D. Barrus, S. G. Hartke, and M. Kumbhat, Graph classes characterized both by forbidden subgraphs and degree sequences, J. Graph Theory 57 (2008), no. 2, 131–148.

[5] M. D. Barrus, S. G. Hartke, and M. Kumbhat, Minimal degree-sequence-forcing triples, in preparation.

[6] M. D. Barrus, S. G. Hartke, and M. Kumbhat, Nonminimal degree-sequence-forcing triples, submitted.

[7] M. D. Barrus and D. B. West, The A_4-structure of a graph, in preparation.

[8] M. D. Barrus and D. B. West, On the degree-associated reconstruction number of a graph, in preparation.

[9] A. Brandstädt and V. B. Le, Split-perfect graphs: characterizations and algorithmic use, SIAM J. Discrete Math. 17 (2004), no. 3, 341–360.

[10] Y. Caro, New results on the independence number, Technical Report, 05-79, Tel Aviv University, 1979.

[11] V. Chvatal, A semi-strong perfect graph conjecture, Ann. Discrete Math. 21 (1984), 279–280.

[12] O. Favaron, M. Mahéo, and J.-F. Saclé, On the residue of a graph, J. Graph Theory 15 (1991), no. 1, 39-64.

[13] P. L. Hammer and B. Simeone, The splittance of a graph, Combinatorica 1 (1981), no. 3, 275–284.

[14] P. J. Kelly, A congruence theorem for trees, Pacific J. Math. 7 (1957), 961–968.

[15] V. B. Le, Bipartite-perfect graphs, Discrete Appl. Math. 127 (2003), no. 3, 581–599.

[16] N. V. R. Mahadev and U. N. Peled, Threshold graphs and related topics. Annals of Discrete Mathematics, 56. North-Holland Publishing Co., Amsterdam, 1995.

[17] S. Ramachandran, On a new digraph reconstruction conjecture, J. Combin. Theory Ser. B 31 (1981), no. 2, 143–149.

[18] S. Ramachandran, Reconstruction number for Ulam's conjecture, Ars Combin. 78 (2006), 289–296.

[19] S. B. Rao, A survey of the theory of potentially P-graphic and forcibly P-graphic degree sequences, Combinatorics and graph theory (Calcutta, 1980), pp. 417–440, Lecture Notes in Math., 885, Springer, Berlin-New York, 1981.

[20] B. Reed, A semistrong perfect graph theorem, J. Combin. Theory Ser. B 43 (1987), no. 2, 223–240.

[21] P. K. Stockmeyer, The falsity of the reconstruction conjecture for tournaments, J. Graph Theory 1 (1977), no. 1, 19-25.

[22] R. I. Tyshkevich, Canonical decomposition of a graph, Dokl. Akad. Nauk BSSR 24 (1980), no. 8, 677–679, 763. In Russian.

[23] R. I. Tyshkevich, Decomposition of graphical sequences and unigraphs, Discrete Math. 220 (2000), no. 1-3, 201–238.

[24] S. M. Ulam, A collection of mathematical problems. Interscience Tracts in Pure and Applied Mathematics, no. 8. Interscience Publishers, New York-London, 1960.

[25] V.K. Wei, A lower bound on the stability number of a simple graph, Technical Report, TM 81-11217-9, Bell Labs, 1981.