Math 231 Project Option #8
Euler's Amazing Formula, or, Those Hyperbolic Cousins

Introducing the formula

The great mathematician Leonhard Euler is one of the greatest mathematicians the world has ever known. By the time he died in 1783, he'd written enough mathematics to fill at least 70 thick volumes (and definitely more—they're *still* publishing new material of his), and much of that while he was blind. Euler's influence on mathematics extends to many areas of mathematics. The Wikipedia entry "List of topics named after Leonhard Euler" contains an astonishing number of items in mathematics and science that are tied by name to this man.

Because of that, if you had to name a formula as "Euler's Formula," it's not at all certain which one you'd pick. Here's one possibility. In 1748, Euler wrote down the following statement:

$$e^{it} = \cos t + i \sin t.$$

The i in the formula is just what you remember it being from algebra—it's the most basic imaginary number, the square root of -1.

In this project we'll examine this formula and a related tangent or two. Let's begin with a few facts.

- Using the fact that $i^2 = 1$, write down the 0th through 8th powers of i. What pattern do you see in the powers?
- Write down the Maclaurin series for the function e^x.
- Substitute $x = it$ into the Maclaurin series, and simplify the powers of i that result.
- Separate the terms of the series that are imaginary (those that still contain i) from those that are real. In other words, write the Maclaurin series for e^{it} as a sum

$$\sum_{k=0}^{\infty}\Big(\qquad\Big)t^n + i\sum_{k=0}^{\infty}\Big(\qquad\Big)t^n.$$

- The two power series that result in the last bullet point should be familiar to you—what functions are they the power series of? How does this all relate to Euler's Formula?

So, knowing what we do about the Maclaurin series for e^x, we can arrive at Euler's Formula. As a side note, let's see what happens when we let $t = \pi$ in Euler's Formula. We find $e^{i\pi} = \cos \pi + i \sin \pi$, and if we simplify and rearrange this, we get the equation

$$e^{i\pi} + 1 = 0,$$

which is a celebrity showcase of *five* wonderful mathematical constants.

Euler's Formula is a powerful tool for working with complex numbers. Since our course this semester doesn't deal with imaginary numbers, though, let's see what it can do when you only work with real numbers.

De Moivre's Theorem

One immediate application of Euler's Formula is a proof of de Moivre's Theorem, which says that

$$(\cos t + i \sin t)^n = \cos nt + i \sin nt. \qquad (*)$$

- How do we know, from Euler's Formula, that the left- and right-hand sides of the equation above are equal? (What third thing are they both equal to, and why?)

De Moivre's Theorem still involves complex numbers, but it has applications to trig identities. Remember the double-angle identities

$$\cos 2t = \cos^2 t - \sin^2 t \qquad \text{and} \qquad \sin 2t = 2\sin t \cos t?$$

Well, we get them by letting $n = 2$ in de Moivre's Theorem (Equation $(*)$). On the left-hand side, we get

$$\begin{aligned}(\cos t + i\sin t)^2 &= (\cos t)^2 + 2(\cos t)(i\sin t) + (i\sin t)^2 \\ &= \cos^2 t + 2i\cos t\sin t + i^2\sin^2 t \\ &= (\cos^2 t - \sin^2 t) + i(2\sin t\cos t).\end{aligned}$$

Notice that in the last line, we use the fact that $i^2 = -1$, and we collect all the real terms (those without an i) together, and we collect all the imaginary terms (those with an i) together.

Now let's see what the right-hand side of de Moivre's Theorem looks like when $n = 2$:

$$\cos 2t + i\sin 2t.$$

If we take the simplified left-hand side and the right-hand side of Equation $(*)$ and put them together, we get

$$(\cos^2 t - \sin^2 t) + i(2\sin t\cos t) = \cos 2t + i\sin 2t.$$

Now look at this formula. The only way that two complex numbers can be equal is if their real parts are equal and their imaginary parts are equal. Therefore, we can write

$$\cos^2 t - \sin^2 t = \cos 2t \qquad \text{and} \qquad 2\sin t\cos t = \sin 2t.$$

Recognize these? These are exactly the double-angle identities!

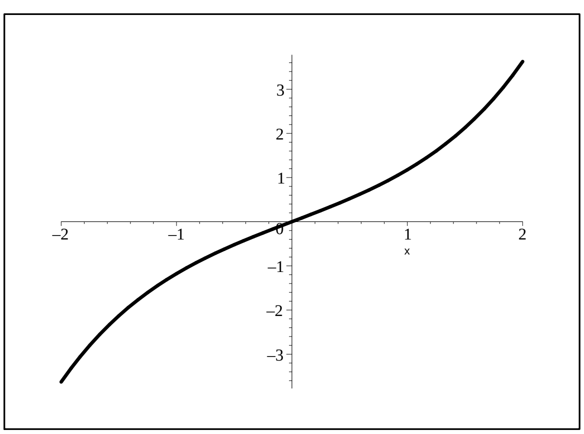
- Using the same technique just demonstrated, use de Moivre's Theorem to come up with identities for $\sin 3t$ and $\cos 3t$, and for $\sin 4t$ and $\cos 4t$.

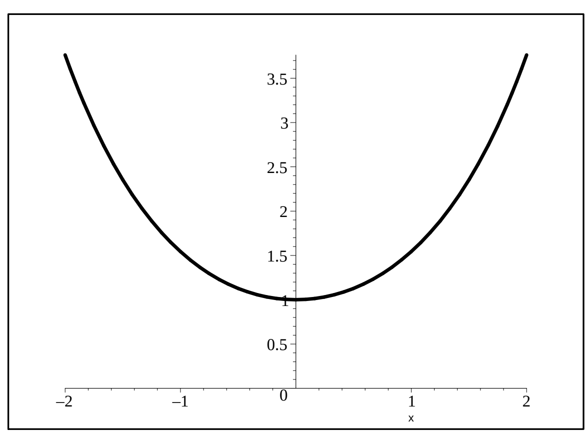
Sines and cosines, hyperbolic or not

You may have wondered at some point why the functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$

the *hyperbolic sine* and *hyperbolic cosine* functions, are called a "sine" or a "cosine" function at all. After all, their graphs, shown below, look nothing at all like the graphs of sine and cosine:





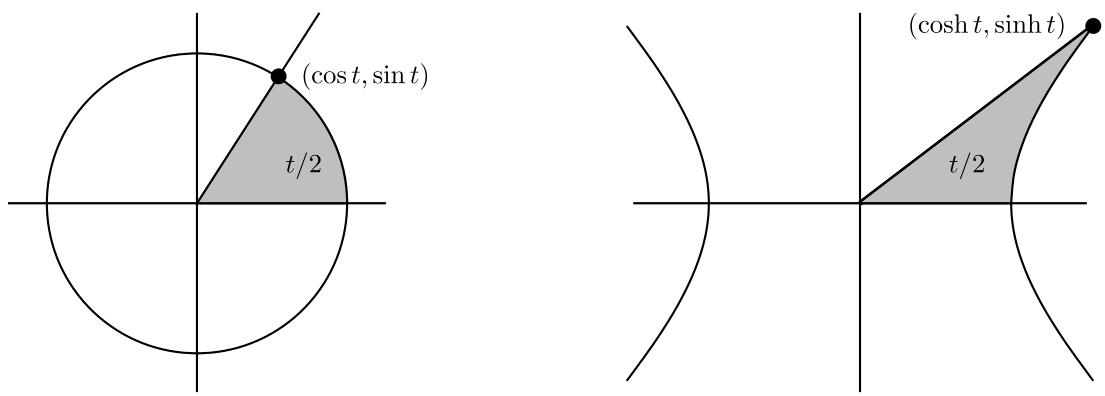
However, there are some connections, which you'll now demonstrate.

- Starting with the Maclaurin series for e^x and e^{-x}, find the Maclaurin series for $\sinh x$ and $\cosh x$.
- What are the Maclaurin series for $\sin x$ and $\cos x$? How do these series compare to those for the functions $\sinh x$ and $\cosh x$?
- Now let's alter the formulas for $\sinh x$ and $\cosh x$ just a bit, so that we get

  $$\frac{e^{ix} - e^{-ix}}{2} \qquad \text{and} \qquad \frac{e^{ix} + e^{-ix}}{2}.$$

  Using Euler's Formula, simplify each of the following expressions as much as you can. You should see yet another connection between $\sin x$ and $\cos x$ and their hyperbolic cousins.

We'll end this project by looking at one more connection between the trigonometric and hyperbolic functions. Recall that the sine and cosine functions can be defined in terms of the unit circle $x^2 + y^2 = 1$. When you draw an angle t centered at the origin, with one side on the positive x-axis, then $\sin t$ is defined as the y-coordinate of the angle's intersection with the unit circle, and $\cos t$ is defined as the x-coordinate of this same point. The circular sector swept out by the angle t has area $t/2$. This is illustrated on the left below.



Let's look now at the "unit hyperbola" $x^2 - y^2 = 1$, pictured on the right above. Again we'll place one side of an angle on the x-axis and open the angle in the counterclockwise direction. When the area bounded by the angle and the hyperbola equals $t/2$, we'll see where the angle intersects the hyperbola, as shown. It turns out that the x-coordinate of the intersection is exactly $\cosh t$, and the y-coordinate is exactly $\sinh t$, just like with the circle and the sine and cosine functions. Let's see why this is.

- Suppose that $x = \cosh t$, $y = \sinh t$. Using the definitions of $\sinh t$ and $\cosh t$, show that $x^2 - y^2 = 1$.
- The last bullet point shows that the parametric curve $x = \cosh t$, $y = \sinh t$ traces out at least part of the unit hyperbola $x^2 - y^2 = 1$. *How much* of the unit hyperbola (which, remember, has two "branches") is traced out as t ranges from $-\infty$ to ∞?
- Suppose now that we don't know the area of the shaded region to begin with. If we join the point $(\cosh T, \sinh T)$ to the origin with a line segment, closing off a "triangular" region whose other boundaries are the x-axis and the hyperbola, as in the picture above, then show that the region enclosed has area $T/2$. (Hint: you don't *have* to use a parametric formula to find the area.)

We've seen, then, yet another reason why these functions are called the hyperbolic *sine* and *cosine* functions: both they and the "normal" sine and cosine functions are defined as places where lines intersect geometric curves which involve x^2, y^2, and 1. Furthermore, the unit hyperbola now gives us a reason for why $\sinh x$ and $\cosh x$ are called *hyperbolic* functions.

So there you have it. Call it a small world, but whether or not it surprises you, the functions e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, and the numbers e, π, i, 0, 1 are all connected! And now you know how.