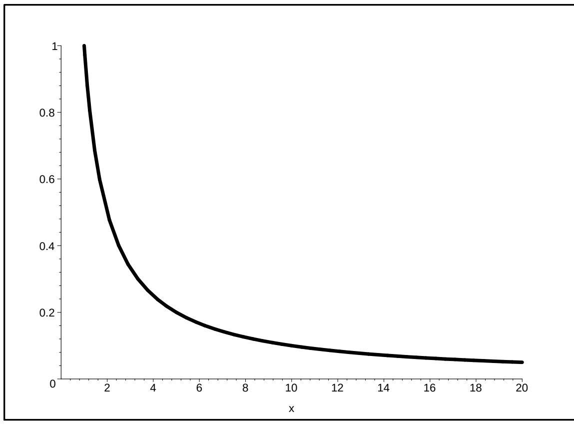
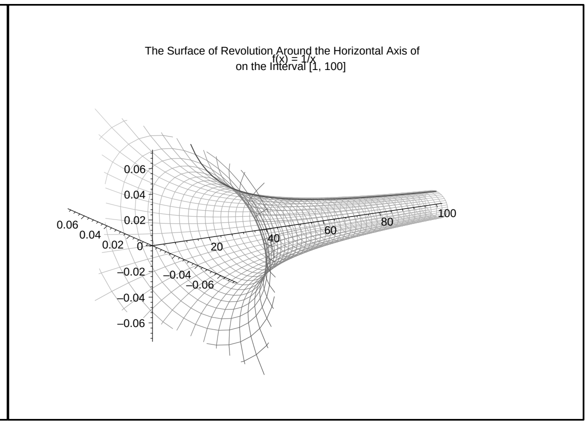
Math 231 Project Option #1
Gabriel's Horn and the Mysteries of Infinity

In the year 1641 the Italian physicist and mathematician Evangelista Torricelli discovered some interesting facts that at first glance present a paradox. Toricelli considered the object formed when the curve

$$y = \frac{1}{x}, \quad x \geq 1$$

is revolved around the x-axis. A surface is swept out that resembles an infinitely long trump, known now as *Toricelli's trumpet* or *Gabriel's Horn*. Shown below are depictions of the curve $y = 1/x$ and the surface of revolution:





Let's start by finding a few things out about Gabriel's Horn. Perform the following activities, and write them up in your project report.

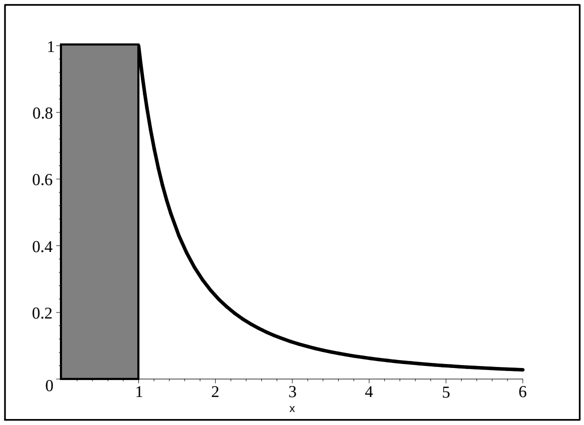
- Set up and evaluate an improper integral representing the volume enclosed by Gabriel's Horn.
- Set up an improper integral representing the surface area of (the inside of) Gabriel's Horn. In this problem and the one above, you may want to refer to Chapter 5 in your text.
- Find the antiderivative of the integrand of the surface area integral. (Hint: Use a trigonometric substitution.) Then evaluate the integral to determine how much surface area Gabriel's Horn has.
- Now look at the surface area integral and use the comparison test for improper integrals to show the same thing you found above about the surface area.
- If you wished to fill Gabriel's Horn with paint, how much would you need?
- What might the average person respond when asked how much paint would be require to paint just the *surface* of Gabriel's Horn? Comment on the apparent paradox.

The seeming paradox illustrated in the questions above caused a philosophical storm in the 17th century. Torricelli stood by his mathematics, while the philosopher Thomas Hobbes took exception to its implications. In a debate with the mathematician John Wallis on this subject, Hobbes is reputed to have said, "To understand this for sense it is not required that a man should be a geometrician or a logician, but that he should be mad."

In this project, you'll attempt to resolve the paradox. Let's start by examining a few other questions dealing with infinity:

- Let's play a game in which we're drawing a rectangle. I name a length for the rectangle, and you name a height. The only rule is that the rectangle with my length and your height must have area equal to 1 square unit. What height will you specify if I make the length 1 unit? 100 units? 1,000,000 units? Is there any length that's just too long—can I choose a length so long that you won't be able to pick a height to make the area exactly 1 unit?

- Next, look at the following figure:



  The curve at the right is $y = 1/x^2$. Show that the area under this curve for $x \geq 1$ is 1 square unit. The shaded rectangle on the right, if the graph were drawn to scale, would actually be a square with its sides all 1 unit long. Notice that the unshaded region has a left edge that is 1 unit long, just like the square. However, the square is only 1 unit wide, while the width of the region under the curve is infinite. How does the unshaded region manage to keep the same area as the square?

- Now we'll look at one more thought experiment. Suppose you have a gallon of lemonade and very precise measuring equipment—no matter how small the quantity of lemonade you need, your equipment can fill a cup with just that much. You're given the responsibility to dole out lemonade to a line of people so long you can't even see the end. In order to make the lemonade last, you always give the first person in line half of what you have left. For example, the first cup you fill will receive 1/2 gallon of lemonade, while the second cup will receive 1/4 gallon (which is half of what's left). The third cup will receive 1/8 gallon, and so on. The question is this: following the procedure described, will you ever run out of lemonade? What if the line turns out to be infinitely long? Would you ever reach a spot in the line where you don't have anything to give?

Now, keeping in mind what you've just discovered in these other problems, let's look back at the problem of painting Gabriel's Horn:

- Clearly, filling the horn with paint accomplishes the task of painting (the inside of) the horn, so the volume you found at the very beginning of the project is enough paint to cover the surface. How about 1 cubic unit of paint? Is that enough to cover the surface, in theory? Is there any volume at all that is too small to cover the horn's surface area?

- Up until now, we've been dealing with a very idealized situation. Explain why in real life, even if we discovered a prototype of Gabriel's Horn floating in outer space, a finite volume of paint would never be enough to paint the outside of it. (Hint: You'll probably want to consider discussing such things as the size of atoms, etc.)