Math 231 Project Option #3
A "Harmonic Sequence"

The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ is an important example in the study of series, because it is a series whose terms approach zero which still somehow manages to diverge. In this project we'll take a look at an interesting sequence which has ties to the harmonic series and alternating harmonic series.

We start by defining the sequence $\{a_k\}_{k=1}^{\infty}$ by

$$a_k = \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k}.$$

For example,

$$a_1 = \frac{1}{2}, \qquad a_2 = \frac{1}{3} + \frac{1}{4}, \qquad a_3 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6}, \qquad \text{and} \quad a_4 = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}.$$

Let's examine the behavior of this sequence.

- Using a calculator or computer, write down the first 10 terms of this series, with each term accurate to at least 4 decimal places.

The sequence does appear to be converging, though it does so pretty slowly. Using a computer algebra system, we find that

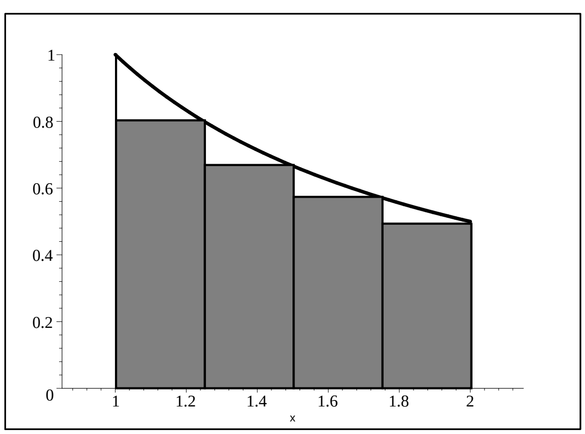
$$a_{100} \approx .6906534305, \qquad a_{1000} \approx .692897243, \qquad a_{10000} \approx .693122181, \qquad \text{and} \quad a_{100000} \approx .69314469.$$

Luckily, we can show that it converges without actually knowing what it converges to. In your project report, do the following:

- Show that each term is at least 1/2.
- Show that each term is no bigger than 1.
- Show that the sequence $\{a_k\}_{k=1}^{\infty}$ is increasing. (Hint: show that $a_{k+1} - a_k$ is always positive.)
- Look at Theorem 1.4 on page 621 of your text. Based on the theorem and what you've shown before, what can you conclude about the sequence?

Based on some numbers and a theorem from the text, we can say that the sequence converges. It's a little unsettling to realize that we don't yet have a good idea of what the sequence $\{a_k\}_{k=1}^{\infty}$ converges *to*. Let's look at that next.

Let's jump to a seemingly unrelated topic. Suppose we want to approximate the area under the curve $y = 1/x$ between $x = 1$ and $x = 2$ by drawing n rectangles of equal width whose right-hand top corners all touch the curve, as illustrated below:



We're going to set up the Riemann sum described.

- First suppose that x_i denotes the x-coordinate of the right-hand end of the ith rectangle. Since there are n rectangles, each rectangle has width $1/n$, and the nth rectangle has its right end at $x_n = 2$. Find a formula for x_i, where i can be anything from 1 to n.
- The height of the ith rectangle is the height of the graph $y = 1/x$ at the right endpoint of the ith rectangle. In other words, the height of the ith rectangle is $1/x_i$. Use this fact to come up with a formula for A_n, the sum of the areas of the rectangles when there are n rectangles.

Now we're going to tie the rectangle areas A_n to the sequence terms a_k:

- Use a little algebra to show that the sequence term

  $$a_k = \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}$$

  is exactly equal to A_k, the total area covered when k rectangles are used to approximate the area under the graph of $y = 1/x$ from $x = 1$ to $x = 2$.
- This means that to see what the sequence $\{a_k\}_{k=1}^{\infty}$ converges to, we only have to see what the Riemann sums approach as we use more and more rectangles—this is exactly the integral $\int_1^2 \frac{1}{x}\,dx$! Find this value, and comment on how closely the sequence terms you computed at the beginning of the project come to this limit.
- Explain how the harmonic series can be thought of as a sum of 1 together with certain of the a_k-terms (explain which ones), and use what we know about the a_k's to give another explanation of why the harmonic series diverges.

Let's change pace a bit and take a look at the alternating harmonic series:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots.$$

- Using a calculator or computer, compute the first 10 partial sums of this series, with each term accurate to at least 4 decimal places.
- Comparing these partial sums with the sequence terms you computed at the beginning of the project, you may notice the startling fact that certain of the partial sums seem to equal certain of the terms a_k! What pattern do you notice? Demonstrate it with examples from the numbers you computed above.

The rest of the project will be showing that this pattern is more than just coincidence.

- Show that for the partial sum S_{2n} of the alternating harmonic series,

  $$S_{2n} = \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k}.$$

  (Hint: you may find it more illuminating to show that

  $$S_{2n} = \sum_{k=1}^{2n} \frac{1}{k} - 2\sum_{k=1}^{n} \frac{1}{2k}$$

  first.)

- Explain why
$$\sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k}.$$
- Explain, from the steps above, why $S_{2n} = a_n$, where a_n is defined as at the start of the project.
- Having proved a connection between the partial sums of the harmonic series and the sequence $\{a_k\}_{k=1}^{\infty}$, explain what value the alternating harmonic series converges to, and why.