Math 231 Project Option #2
Differential Equations and the Wonderful Transform of Laplace

In Section 7.1 we briefly touched on the idea of a *differential equation*, i.e., an equation involving the derivatives of an unknown function $y(x)$. Solving for the function $y(x)$ can require some clever manipulation of the differential equation. For example, you might look back in your textbook to see how the authors started with the differential equation

$$y'(x) - ky(x) = 0$$

and arrived at the solution

$$y(x) = Ae^{kx}.$$

So how do you solve other differential equations? For instance, how would you solve something like the following:

$$y'(x) + 2y(x) = \sin x, \qquad y(0) = 0?$$

The French mathematician Pierre-Simon de Laplace (1749-1827) developed ideas of Leonhard Euler (1707-1783) and Joseph Louis Lagrange (1736-1813) and presented a transformation on a function that seems to help. Later, this transformation was used by Oliver Heaviside (1850-1925) to change differential equations into equations from algebra which are much easier to solve.

The change Laplace applied to functions, which nowadays we call the *Laplace transform* of a function $f(x)$, is denoted by $\mathcal{L}\{f(x)\}$ or by $Y(s)$, and it's defined by

$$Y(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx.$$

It's not immediately obvious why this should be interesting to us, or how Laplace came up with it, but we'll see below that it's a pretty slick operation for turning differential equations into algebra problems.

First, let's get some exposure to the Laplace transform of a function. As an example, let's derive the Laplace transform of the function $f(x) = 1$:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-sx} \cdot 1\, dx = \lim_{b\to\infty} \left.\frac{e^{-sx}}{-s}\right|_0^b = \lim_{b\to\infty} \frac{e^{-sb}}{-s} + \frac{1}{s} = \frac{1}{s}.$$

You'll notice that in the example, we treated s as a constant throughout the integration. Notice also that the improper integral only converges if $s > 0$ (in taking the limit as b approaches infinity, we need the e^{-sb} term to go to zero, so $-sb$ has to be negative). We say, then, that the Laplace transform of 1 is $1/s$ for $s > 0$.

Try your hand at a few of these, by finding the Laplace transforms of the following functions. In the write-up of your project, be sure to show all steps of your work.

- Find $\mathcal{L}\{x\}$ and $\mathcal{L}\{x^2\}$.
- Find $\mathcal{L}\{e^{ax}\}$, where a is a constant.
- Find $\mathcal{L}\{\sin ax\}$, where a is a constant.

There are a few rules which simplify finding the Laplace transform of more complex functions. For example, you can factor constants outside of Laplace transforms, i.e., if c is a constant, then

$$\mathcal{L}\{c \cdot f(x)\} = c \cdot \mathcal{L}\{f(x)\}.$$

So, for example, the Laplace transform of any constant function $f(x) = c$ can be found as follows:

$$\mathcal{L}\{c\} = \mathcal{L}\{c \cdot 1\} = c \cdot \mathcal{L}\{1\} = c \cdot \frac{1}{s} = \frac{c}{s},$$

for $s > 0$. The proof is pretty straightforward, just using the definition of the Laplace transform (you might try on your own to see why the rule works). Another rule which comes in handy is the following:

$$\mathcal{L}\left\{f(x)+g(x)\right\} = \mathcal{L}\left\{f(x)\right\} + \mathcal{L}\left\{g(x)\right\}.$$

Now let's see why the Laplace transform is so useful for differential equations. Let's suppose that $Y(s)$ is the Laplace transform of $f(x)$, i.e.,

$$Y(s) = \int_0^\infty e^{-sx} f(x)\,dx.$$

What's the Laplace transform of $f'(x)$? Notice that

$$\begin{aligned}
\int_0^\infty e^{-sx} f'(x)\,dx &= \lim_{b\to\infty} \int_0^b e^{-sx} f'(x)\,dx \\
&= \lim_{b\to\infty} \left[e^{-sx} f(x)\Big|_0^b + s\int_0^b e^{-sx} f(x)\,dx \right] \qquad \text{(int. by parts, } u = e^{-sx},\ dv = f'(x)\,dx\text{)} \\
&= \lim_{b\to\infty} \left[e^{-sb} f(b) - f(0) + s\int_0^b e^{-sx} f(x)\,dx \right] \\
&= \left(\lim_{b\to\infty} \frac{f(b)}{e^{sb}} \right) - f(0) + s\int_0^\infty e^{-sx} f(x)\,dx \\
&= -f(0) + sY(s),
\end{aligned}$$

for $s > 0$, as long as $\lim_{b\to\infty} f(b)/e^{sb} = 0$ (that's true for most functions $f(x)$ that we're interested in, so we'll forget about that condition). Did you see how the integration by parts produced an integral that we recognized as $Y(s)$? Thus if $Y(s) = \mathcal{L}\left\{f(x)\right\}$, then

$$\mathcal{L}\left\{f'(x)\right\} = sY(s) - f(0).$$

- Use this rule and what you know about $\mathcal{L}\left\{\sin ax\right\}$ to find $\mathcal{L}\left\{\cos ax\right\}$.
- Show, using integration by parts (and showing all your steps), that

$$\mathcal{L}\left\{f''(x)\right\} = s^2 Y(s) - sf(0) - f'(0).$$

So here's how the Laplace transform is useful. Say you're given a differential equation like

$$y'(x) + y(x) = \sin x, \qquad y(0) = 1.$$

If we apply the Laplace transform to both sides of the equation (and let $Y(s)$ denote $\mathcal{L}\left\{y(x)\right\}$), we find

$$\begin{aligned}
\mathcal{L}\left\{y'(x)+y(x)\right\} &= \mathcal{L}\left\{\sin x\right\}; \\
\mathcal{L}\left\{y'(x)\right\} + \mathcal{L}\left\{y(x)\right\} &= \mathcal{L}\left\{\sin x\right\}; \\
\left[sY(s) - y(0)\right] + Y(s) &= \frac{1}{s^2+1}; \\
(s+1)Y(s) - 1 &= \frac{1}{s^2+1}; \\
Y(s) &= \frac{1}{s+1}\left(\frac{1}{s^2+1} + 1\right);
\end{aligned}$$

and finally

$$Y(s) = \frac{s^2+2}{(s+1)(s^2+1)}.$$

At this point we don't know what $y(x)$ is, but we've been able to solve for $Y(s)$, so we know what the Laplace transform of $y(x)$ is. Maybe if we had a table of Laplace transforms, we'd be able to look at the lineup and see what $y(x)$ is. Below is a simple table of Laplace transforms (you can fill in the blanks with the answers you've already found).

$f(x)$	$\mathcal{L}\{f(x)\}$
1	$\frac{1}{s}$
x	
x^2	
e^{ax}	
$\sin ax$	
$\cos ax$	
$e^{bx}\sin ax$	$\frac{a}{(s-b)^2+a^2}$
$e^{bx}\cos ax$	$\frac{s-b}{(s-b)^2+a^2}$

Now in the problem we're attempting to solve, we have

$$Y(s) = \frac{s^2+2}{(s+1)(s^2+1)}.$$

You'll notice that the fraction doesn't look like any of the Laplace transforms in the table. In particular, if you were to expand the denominator, it'd be a cubic function, where everything in the table has a linear or a quadratic denominator. Can we rewrite $Y(s)$ as a combination of functions with smaller denominators, so as to use the table? You guessed it! We need to find the partial fraction decomposition of $Y(s)$. Doing so, we arrive at

$$Y(s) = \frac{s^2+2}{(s+1)(s^2+1)} = \frac{3/2}{s+1} + \frac{(-1/2)s+1/2}{s^2+1}.$$

Rewriting this just a bit so that it looks more like the table entries, we get

$$Y(s) = \frac{3}{2}\cdot\frac{1}{s-(-1)} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1}.$$

Looking at the table (including the entries you've filled in), we see that if

$$y(x) = \frac{3}{2}e^{-x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x,$$

Then $y(x)$ will have $Y(s)$ as its Laplace transform. That means that this choice of $y(x)$ is the solution to the differential equation. Let's check the answer in the original differential equation. Is

$$y'(x) + y(x) = \sin x?$$

Well,

$$y'(x) + y(x) = \left(-\frac{3}{2}e^{-x} + \frac{1}{2}\sin x + \frac{1}{2}\cos x\right) + \left(\frac{3}{2}e^{-x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x\right) = \sin x,$$

so that works out. Furthermore,

$$y(0) = \frac{3}{2}e^0 - \frac{1}{2}\cos 0 + \frac{1}{2}\sin 0 = 1.$$

So yes! The Laplace transform technique gave us the differential equation's solution!

Now here's your chance to use the Laplace transform. Using the facts you've read about and worked out above, solve the following two differential equations. Show all your steps.

- Find $y(x)$, where

$$y'(x) + 2y(x) = 2, \qquad y(0) = 1.$$

- Find $y(x)$, where

$$y' + 2y = e^x, \qquad y(0) = 1.$$