

The Math 231 first mock exam from last year

While going through some papers and files, I found the following problems, which I put together last year for a Math 231 merit section. The test on February 11 will cover a bit more material than this, but these questions can get you started on studying. They're about the level of difficulty you should expect on the test. Solutions will be posted online shortly, and a second mock exam will be given during class on Friday, February 8. Here you go:

Work the following practice test as if it were your actual midterm. No books, notes, or calculators are allowed. Good luck!

1. Evaluate the integrals:

(a) $\int \frac{t}{\sqrt{1-9t^2}} dt$

(b) $\int \frac{\sec^2 \theta}{1 + \tan \theta} d\theta$

2. Find the following integrals:

(a) $\int x e^{2x} dx$

(b) $\int e^{2x} \cos 4x dx$

(c) $\int x^3 \sin x^2 dx$

3. Find the following integrals:

(a) $\int \sin 3x \cos 5x \, dx$

(b) $\int \frac{1}{\cos^4 2x} \, dx$

4. Find the integral:

$$\int \frac{x^3}{x^2 + x - 6} dx$$

5. Find the integral:

$$\int \frac{\sqrt{4x^2 - 1}}{x} dx$$

Solutions

1. Evaluate the integrals:

$$(a) \quad \int \frac{t}{\sqrt{1-9t^2}} dt$$

Solution. Noticing the t^2 down below and the $t dt$ up top, we make the substitution $u = 1 - 9t^2$. This gives us

$$\begin{aligned} \int \frac{t}{\sqrt{1-9t^2}} dt &= -\frac{1}{18} \int u^{-1/2} du \\ &= -\frac{1}{9} u^{1/2} + C \\ &= -\frac{1}{9} \sqrt{1-9t^2} + C \end{aligned}$$

$$(b) \quad \int \frac{\sec^2 \theta}{1 + \tan \theta} d\theta$$

Solution. Noticing that $\sec^2 \theta$ is the derivative of $\tan \theta$, we make the substitution $u = 1 + \tan \theta$, which gives

$$\begin{aligned} \int \frac{\sec^2 \theta}{1 + \tan \theta} d\theta &= \int \frac{du}{u} \\ &= \ln |u| + C \\ &= \ln |1 + \tan \theta| + C \end{aligned}$$

2. Find the following integrals:

$$(a) \quad \int x e^{2x} dx$$

Solution. We use integration by parts, with $u = x$ and $dv = e^{2x} dx$:

$$\begin{aligned} \int x e^{2x} dx &= \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C \end{aligned}$$

$$(b) \int e^{2x} \cos 4x \, dx$$

Solution. We use integration by parts, letting $u = e^{2x}$ and $dv = \cos 4x \, dx$. This gives us

$$\int e^{2x} \cos 4x \, dx = \frac{1}{4}e^{2x} \sin 4x - \frac{1}{2} \int e^{2x} \sin 4x \, dx.$$

We apply integration by parts to the second integral, with $u = e^{2x}$ and $dv = \sin 4x$. (Careful here! If you let $u = \sin 4x$ and $dv = e^{2x} \, dx$, you'll just be undoing your previous integration by parts, and you'll get $0 = 0$.) We get

$$\begin{aligned} \int e^{2x} \cos 4x \, dx &= \frac{1}{4}e^{2x} \sin 4x - \frac{1}{2} \int e^{2x} \sin 4x \, dx \\ &= \frac{1}{4}e^{2x} \sin 4x - \frac{1}{2} \left[-\frac{1}{4}e^{2x} \cos 4x + \frac{1}{2} \int e^{2x} \cos 4x \, dx \right] \\ &= \frac{1}{4}e^{2x} \sin 4x + \frac{1}{8}e^{2x} \cos 4x - \frac{1}{4} \int e^{2x} \cos 4x \, dx. \end{aligned}$$

Notice now that the integral we want to solve for also appears on the right. We move it over to the left-hand side and solve for it:

$$\begin{aligned} \frac{5}{4} \int e^{2x} \cos 4x \, dx &= \frac{1}{4}e^{2x} \sin 4x + \frac{1}{8}e^{2x} \cos 4x + C; \\ \int e^{2x} \cos 4x \, dx &= \frac{1}{5}e^{2x} \sin 4x + \frac{1}{10}e^{2x} \cos 4x + C. \end{aligned}$$

$$(c) \int x^3 \sin x^2 \, dx$$

Solution. Having an x^2 inside a sine is an ugly thing, as far as integration is concerned, so we try a substitution first; let $w = x^2$. We get

$$\int x^3 \sin x^2 \, dx = \frac{1}{2} \int w \sin w \, dw.$$

Now we apply integration by parts, with $u = w$ and $dv = dw$:

$$\begin{aligned} \int x^3 \sin x^2 \, dx &= \frac{1}{2} \int w \sin w \, dw \\ &= \frac{1}{2} \left[-w \cos w + \int \cos w \, dw \right] \\ &= -\frac{1}{2}w \cos w + \frac{1}{2} \sin w + C \\ &= -\frac{1}{2}x^2 \cos x^2 + \frac{1}{2} \sin x^2 + C. \end{aligned}$$

3. Find the following integrals:

(a) $\int \sin 3x \cos 5x \, dx$

Solution. This one can be integrated quickly by using the trig identity

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)].$$

However, we can also find the integral by performing integration by parts twice.

$$\begin{aligned} \int \sin 3x \cos 5x \, dx &= \frac{1}{5} \sin 3x \sin 5x - \frac{3}{5} \int \cos 3x \sin 5x \, dx \\ &= \frac{1}{5} \sin 3x \sin 5x - \frac{3}{5} \left[-\frac{1}{5} \cos 3x \cos 5x + \frac{3}{5} \int \sin 3x \cos 5x \, dx; \right] \\ &= \frac{1}{5} \sin 3x \sin 5x + \frac{3}{25} \cos 3x \cos 5x - \frac{9}{25} \int \sin 3x \cos 5x \, dx; \end{aligned}$$

$$\frac{34}{25} \int \sin 3x \cos 5x \, dx = \frac{1}{5} \sin 3x \sin 5x + \frac{3}{25} \cos 3x \cos 5x + C;$$

$$\int \sin 3x \cos 5x \, dx = \frac{5}{34} \sin 3x \sin 5x + \frac{3}{34} \cos 3x \cos 5x + C.$$

(b) $\int \frac{1}{\cos^4 2x} \, dx$

Solution. Since $1/\cos \theta = \sec \theta$, we rewrite the integral as

$$\int \sec^4 2x \, dx.$$

Now $\sec^2 \theta$ is the derivative of $\tan \theta$, so we split a $\sec^2 2x$ off from the integrand and use an identity on the rest of the integral:

$$\begin{aligned} \int \frac{1}{\cos^4 2x} \, dx &= \int \sec^4 2x \, dx \\ &= \int \sec^2 2x \sec^2 2x \, dx \\ &= \int (1 + \tan^2 2x) \sec^2 2x \, dx. \end{aligned}$$

We make the substitution $u = \tan 2x$, which gives

$$\begin{aligned}\int \frac{1}{\cos^4 2x} dx &= \int (1 + \tan^2 2x) \sec^2 2x dx \\ &= \frac{1}{2} \int (1 + u^2) du \\ &= \frac{1}{2} \left(u + \frac{1}{3} u^3 \right) + C \\ &= \frac{1}{2} \tan 2x + \frac{1}{6} \tan^3 2x + C.\end{aligned}$$

4. Find the integral:

$$\int \frac{x^3}{x^2 + x - 6} dx$$

Solution. This is a rational function where the numerator has larger degree than the denominator, so we use long division first, as follows:

$$\begin{array}{r} x^2 + x - 6 \overline{) x^3 } \\ \underline{-x^3 - x^2 + 6x} \\ -x^2 + 6x \\ \underline{x^2 + x - 6} \\ 7x - 6 \end{array}$$

i.e.,

$$\frac{x^3}{x^2 + x - 6} = x - 1 + \frac{7x - 6}{x^2 + x - 6}.$$

We now find the partial fraction decomposition of the fraction:

$$\begin{aligned}\frac{7x - 6}{x^2 + x - 6} &= \frac{7x - 6}{(x + 3)(x - 2)} \\ &= \frac{A}{x + 3} + \frac{B}{x - 2};\end{aligned}$$

$$7x - 6 = A(x - 2) + B(x + 3);$$

$$-27 = -5A \quad (\text{using } x = -3)$$

$$A = 27/5;$$

$$8 = 5B \quad (\text{using } x = 2)$$

$$B = 8/5;$$

$$\frac{7x - 6}{x^2 + x - 6} = \frac{27/5}{x + 3} + \frac{8/5}{x - 2}.$$

We now find the integral:

$$\begin{aligned} \int \frac{x^3}{x^2 + x - 6} dx &= \int \left(x - 1 + \frac{27/5}{x+3} + \frac{8/5}{x-2} \right) dx \\ &= \frac{1}{2}x^2 - x + \frac{27}{5} \ln|x+3| + \frac{8}{5} \ln|x-2| + C. \end{aligned}$$

5. Find the integral:

$$\int \frac{\sqrt{4x^2 - 1}}{x} dx$$

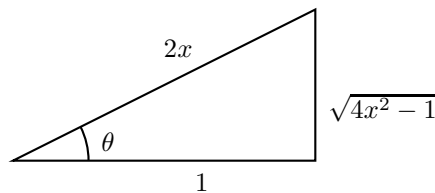
Solution. Because the $4x^2 - 1$ up top fits the form $u^2 - a^2$, we make the substitution $u = a \sec \theta$; in this case that means we set

$$2x = \sec \theta.$$

Then $2 dx = \sec \theta \tan \theta d\theta$, and our integral becomes

$$\begin{aligned} \int \frac{\sqrt{4x^2 - 1}}{x} dx &= \int \frac{\sqrt{\sec^2 \theta - 1}}{\frac{1}{2} \sec \theta} \frac{1}{2} \sec \theta \tan \theta d\theta \\ &= \int \tan^2 \theta d\theta \\ &= \int (\sec^2 \theta - 1) d\theta \\ &= \tan \theta - \theta + C. \end{aligned}$$

Since our original substitution was in terms of $\sec \theta$, we'll draw a triangle to find out what the other trig functions in the answer should equal. The original substitution was $2x = \sec \theta$, so we draw the following triangle and use the Pythagorean theorem to find the length of the side opposite θ :



Then

$$\begin{aligned} \int \frac{\sqrt{4x^2 - 1}}{x} dx &= \tan \theta - \theta + C \\ &= \sqrt{4x^2 - 1} - \sec^{-1}(2x) + C. \end{aligned}$$