Some practice for the second midterm

The following is a collection of problems gleaned from the mock exams from last year and this year. I can't say that this is a good model of the mock exam you'll see Friday, but hopefully it'll get you started in studying.

1. Determine whether or not the sequences below converge, and find the limits of those that do:

$$\text{(a)} \quad a_k = k \cos \pi k$$

$$\text{(b)} \quad a_k = \frac{\ln 2k}{\ln 3k}$$

$$\text{(c)} \quad a_k = \frac{k^2 - k + 7}{2k^2 + 4}$$

$$\text{(d)} \quad a_k = \frac{(-1)^k (\ln k)^2}{k}$$

2. Determine whether the given sums below converge or diverge. If they converge, find their sum.

$$\text{(a)} \quad \sum_{n=2}^{\infty} (-1)^n \left(\frac{e}{3}\right)^n$$

$$\text{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\text{(c)} \quad \sum_{n=1}^{\infty} 2^{1/n}$$

$$\text{(d)} \quad \sum_{k=1}^{\infty} 3^{k-2} 2^{-2k-1}$$

3. Determine whether or not the following series converge.

(a) $$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

(b) $$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

(c) $$\sum_{n=1}^{\infty} \frac{1}{17n-13}$$

4. Determine whether or not the following series converge. Name any rules you use.

(a) $$\sum_{n=1}^{\infty} \frac{\sin^2(1/n)}{n^2}$$

(b) $$\sum_{n=1}^{\infty} \frac{n!n^2}{(2n)!}$$

(c) $$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$$

5. Determine whether each of the following series converges absolutely, converges conditionally, or diverges:

(a) $$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$$

(b) $$\sum_{n=3}^{\infty} \frac{(-1)^n}{n \ln n}$$

(c) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

(d) $$\sum_{n=1}^{\infty} \frac{(-1)^n n \ln n}{n}$$

6. Examine the alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}.$$

(a) Show that the series converges conditionally.

(b) Give two numbers that the series' limit is guaranteed to be between.

(c) Suppose I wanted to find this series sum to an accuracy of at least $1/1000$ (i.e., I want to ensure that the Nth partial sum is no more than $1/1000$ away from the series sum). How big should N be to ensure this?

7. (a) State the Root Test for the convergence/divergence of a series.

(b) Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n$ **converges or diverges.**

Some practice for the second midterm – Solutions

1. Determine whether or not the sequences below converge, and find the limits of those that do:

$$(\mathbf{a}) \quad a_k = k \cos \pi k$$

Solution. The sequence starts out as $-1, 2, -3, 4, \ldots$. It clearly doesn't converge.

$$(\mathbf{b}) \quad a_k = \frac{\ln 2k}{\ln 3k}$$

Solution. We use l'Hospital's rule to find that

$$\lim_{x\to\infty} \frac{\ln 2x}{\ln 3x} = \lim_{x\to\infty} \frac{2/(2x)}{3/(3x)} = \lim_{x\to\infty} 1 = 1,$$

so the sequence converges to 1.

$$(\mathbf{c}) \quad a_k = \frac{k^2 - k + 7}{2k^2 + 4}$$

Solution. We again use l'Hospital's rule:

$$\lim_{x\to\infty} \frac{x^2 - x + 7}{2x^2 + 4} = \lim_{x\to\infty} \frac{2x - 1}{4x} = \lim_{x\to\infty} \frac{2}{4} = \frac{1}{2}.$$

Thus, the sequence converges to $1/2$.

$$(\mathbf{d}) \quad a_k = \frac{(-1)^k (\ln k)^2}{k}$$

Solution. We ignore the sign of the terms for a moment and see what $|a_k|$ approaches, using l'Hospital's rule:

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2(\ln x)/x}{1} = \lim_{x\to\infty} \frac{2/x}{1} = 0.$$

Since $|a_k|$ approaches zero, the sequence a_k converges to 0.

2. Determine whether the given sums below converge or diverge. If they converge, find their sum.

$$\textbf{(a)} \quad \sum_{n=2}^{\infty} (-1)^n \left(\frac{e}{3}\right)^n$$

Solution. We can rewrite the series as $\sum_{n=2}^{\infty} \left(-\frac{e}{3}\right)^n$ and see that it is a geometric series with a ratio whose absolute value is less than 1 (since $e < 3$). Therefore, the series converges, and to find its value we can either rewrite the series as

$$\sum_{n=2}^{\infty} (-1)^n \left(\frac{e}{3}\right)^n = \left(\frac{e}{3}\right)^2 - \left(\frac{e}{3}\right)^3 + \left(\frac{e}{3}\right)^4 - \cdots = \sum_{n=0}^{\infty} \left(\frac{e}{3}\right)^2 \cdot \left(-\frac{e}{3}\right)^n$$

and use the $a/(1-r)$ formula to compute the value as

$$\frac{(e/3)^2}{1-(-e/3)} = \frac{e^2}{9+3e},$$

or we can use the $a/(1-r)$ formula without rewriting the sum, and then just subtract off the $n = 0$ and $n = 1$ terms:

$$\begin{aligned}
\sum_{n=2}^{\infty} (-1)^n \left(\frac{e}{3}\right)^n &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{e}{3}\right)^n - (-1)^0 \left(\frac{e}{3}\right)^0 - (-1)^1 \left(\frac{e}{3}\right)^1 \\
&= \frac{1}{1-(-e/3)} - 1 + \frac{e}{3} \\
&= \frac{e^2}{9+3e}.
\end{aligned}$$

$$\textbf{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Solution. This series may be a telescoping series. We write out the form of the partial fraction decomposition,

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1},$$

find that $1 = A(n+1) + Bn$, and plug in $n = -1$ and $n = 0$ to find that $A = 1$ and $B = -1$, so

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Writing out the Nth partial sum of the series in this broken-down form, we get

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n(n+1)} &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\
&= \frac{1}{1} - \frac{1}{N+1}.
\end{aligned}$$

It's clear that as N tends to infinity, the series converges to 1.

$$\textbf{(c)} \quad \sum_{n=1}^{\infty} 2^{1/n}$$

Solution. The series' terms have a limit

$$\lim_{n\to\infty} 2^{1/n} = 2^0 = 1,$$

which does not equal zero, so the kth-term test for divergence tells us that the series diverges.

$$\textbf{(d)} \quad \sum_{k=1}^{\infty} 3^{k-2}2^{-2k-1}$$

Solution. We rewrite the series as

$$\sum_{k=1}^{\infty} 3^{k-2}2^{-2k-1} = \sum_{k=1}^{\infty} \frac{3^k}{3^2 \cdot (2^2)^k \cdot 2} = \sum_{k=1}^{\infty} \frac{1}{18} \cdot \left(\frac{3}{4}\right)^k,$$

which is a geometric series, and since $|3/4| < 1$, we know that the series converges. The sum can be found in one of two ways, as shown in part (a), and whichever way is used should yield $1/6$ as the value of the sum.

3. Determine whether or not the following series converge.

$$\textbf{(a)} \quad \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Solution. The series converges, because

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$

and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the comparison test implies that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges as well. (Alternatively, we could also show that the series converges by using the integral test.)

$$\textbf{(b)} \quad \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Solution. We apply the integral test, since the function $f(x) = 1/[x(\ln x)^2]$ is continuous and decreasing. Making the substitution $u = \ln x$, we find

$$\int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C,$$

so

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} -\frac{1}{\ln b} + \frac{1}{\ln 2} = \frac{1}{\ln 2},$$

and we see that the integral converges. The integral test tells us that the series converges as well.

$$\textbf{(c)} \quad \sum_{n=1}^{\infty} \frac{1}{17n-13}$$

Solution. We use the limit comparison test with the harmonic series. Since

$$\lim_{n\to\infty} \frac{1/(17n-13)}{1/n} = \lim_{n\to\infty} \frac{n}{17n-13} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{17-13/n} = \frac{1}{17},$$

and $1/17 > 0$, the limit comparison test tells us that the given series behaves in the same way that the harmonic series does. Since the harmonic series diverges, the series we were given does, too.

4. Determine whether or not the following series converge. Name any rules you use.

$$\textbf{(a)} \quad \sum_{n=1}^{\infty} \frac{\sin^2(1/n)}{n^2}$$

Solution. Since

$$\frac{\sin^2(1/n)}{n^2} < \frac{1}{n^2},$$

and since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the comparison test tells us that the given series converges, too.

$$\textbf{(b)} \quad \sum_{n=1}^{\infty} \frac{n!n^2}{(2n)!}$$

Solution. We apply the ratio test. We find that

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)!(n+1)^2}{(2(n+1))!} \cdot \frac{(2n)!}{n!n^2} \\
&= \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{(n+1)^2}{n^2} \frac{(2n)!}{(2n+2)!} \\
&= \lim_{n\to\infty} \frac{n+1}{1} \frac{(n+1)^2}{n^2} \frac{1}{(2n+1)(2n+2)} \\
&= \lim_{n\to\infty} \frac{(n+1)^3}{n^2(2n+1)(2n+2)} \\
&= \lim_{n\to\infty} \frac{n^3+3n^2+3n+1}{4n^4+6n^3+2n^2} \\
&= \lim_{n\to\infty} \frac{3n^2+6n+3}{16n^3+18n^2+4n} \\
&= \lim_{n\to\infty} \frac{6n+6}{48n^2+36n+4} \\
&= \lim_{n\to\infty} \frac{6}{96n+36} \\
&= 0,
\end{aligned}$$

so by the ratio test the series converges.

$$\textbf{(c)} \quad \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$$

Solution. We apply the limit comparison test with the harmonic series. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1/(n+\sqrt{n})}{1/n} &= \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} \\
&= \lim_{n\to\infty} \frac{1}{1+\sqrt{n}/n} \\
&= 1,
\end{aligned}$$

so the limit comparison test tells us that the given series behaves like the harmonic series, which diverges.

5. Determine whether each of the following series converges absolutely, converges conditionally, or diverges:

$$\textbf{(a)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$$

Solution. This is an alternating series, so we'll use the alternating series test to see if it converges. Note that

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0,$$

and

$$\frac{d}{dx}\frac{\ln x}{\sqrt{x}} = \frac{\frac{1}{x}\cdot\sqrt{x} - \frac{1}{2\sqrt{x}}\cdot \ln x}{x} = \frac{2-\ln x}{2x^{3/2}},$$

which is negative, eventually, so the function $(\ln x)/\sqrt{x}$ is decreasing, which means that $a_{n+1} \leq a_n$ in the series. The alternating series test now implies that the series converges.

To see whether it converges absolutely or only conditionally, we now see if the series $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$ converges. (This is the series we get if we take the absolute value of the terms, which has the effect of just ignoring the sign given by the $(-1)^n$.) We use the comparison test: note that

$$\frac{\ln n}{\sqrt{n}} > \frac{1}{\sqrt{n}},$$

and since $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges (it's a p-series with $p \leq 1$), we know that $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$ diverges as well. Therefore, the original series $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$ converges conditionally.

$$\textbf{(b)} \quad \sum_{n=3}^{\infty} \frac{(-1)^n}{n \ln n}$$

Solution. We check the conditions of the alternating series test. It's clear that

$$\lim_{n\to\infty} \frac{1}{n \ln n} = 0 \qquad \text{and} \qquad \frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n},$$

so the alternating series test shows us that this series converges. We now check to see if it converges absolutely or conditionally. The series $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$ diverges by the integral test, so the original series converges conditionally.

$$\textbf{(c)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

Solution. Applying the alternating series test again, we find that

$$\lim_{n\to\infty} \frac{1}{n^3} \qquad \text{and} \qquad \frac{1}{(n+1)^3} < \frac{1}{n^3},$$

so the test tells us that the series converges. Since the absolute-value series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (it's a p-series with $p > 1$), the series in the problem converges absolutely.

$$\textbf{(d)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n \ln n}{n}$$

Solution. We find that

$$\lim_{n\to\infty} \frac{n \ln n}{n} = \lim_{n\to\infty} \ln n = \infty,$$

so the sequence terms do not approach zero. The kth-term test for divergence tells us that this series diverges.

6. Examine the alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}.$$

(a) Show that the series converges conditionally.

Solution. The series satisfies the conditions of the alternating series test, since

$$\lim_{k\to\infty} \frac{1}{k} = 0 \qquad \text{and} \qquad \frac{1}{k+1} < \frac{1}{k}.$$

Therefore, the series converges. Note, however, that taking the absolute value of each term yields the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$, which diverges, so $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges conditionally.

(b) Give two numbers that the series' limit is guaranteed to be between.
Solution. If a series satisfies the alternating series test, its limit is guaranteed to be between any two consecutive partial sums. This means, for example, that the limit of the series should be $S_1 = 1$ and $S_2 = 1 - 1/2 = 1/2$.

(c) Suppose I wanted to find this series sum to an accuracy of at least $1/1000$ (i.e., I want to ensure that the Nth partial sum is no more than $1/1000$ away from the series sum). How big should N be to ensure this?
Solution. If a series $\sum(-1)^{k+1}a_k$ satisfies the alternating series test and has limit L, then we know that

$$|S_N - L| \leq a_{N+1}.$$

(This says that the distance between the Nth partial sum and the series limit is no more than the size of the next term to be added; for an explanation of this, see pages 652-653 of your text.) We want to make sure that

$$|S_N - L| \leq 1/1000.$$

One way to guarantee this is to make $a_{N+1} \leq 1/1000$. This requires that

$$\frac{1}{N+1} \leq \frac{1}{1000}, \qquad \text{or, solving for } N, \qquad N \geq 999.$$

Thus N should be at least 999.

7. (a) State the Root Test for the convergence/divergence of a series.
Solution. See page 661 of your text.

(b) Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n$ converges or diverges.
Solution. We don't *have* to apply the root test here (the ratio test is doable, as well), as is a comparison test with a convergent geometric series, but the root test is perhaps the easiest test for this series. We take the limit of the nth root of the series term:

$$\lim_{n\to\infty} \left[\left(\frac{2}{n}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

Since the limit is less than 1, the root test says that the series converges.