Selected answers — Merit Worksheet #5

1. (a) A u-substitution with $u = 1 - x$ works, as does integration by parts twice (let $u = x^2$ and $dv = 1/\sqrt{1-x}$ the first time, and $u = x$ and $dv = \sqrt{1-x}$ the second time); one can also set $x = \sin^2\theta$ and do a trigonometric substitution (solving for θ gives $\theta = \sin^{-1}\sqrt{x}$).

   (b) One approach is to use u-substitution with $u = \ln x$; another is to do integration by parts once with $u = 1/(\ln x)^3$, $dv = 1/x\ dx$, and then solve for the original integral.

   (c) A u-substitution with $u = \ln x$ works again; another way is to do integration by parts with $u = \ln x$, $dv = 1/x\ dx$, and then solve for the original integral. A third way would be to do integration by parts with $u = 1/x$ and $dv = \ln x\ dx$; doing this requires you to know that $\int \ln x\ dx = x\ln x - x + C$, but it works.

   (d) One way: do a u-substitution with $u = \sin x$, or else with $u = \cos x$. A second way: Do integration by parts with $u = \sin x$, $dv = \cos x\ dx$ (or vice versa), and then solve for the original integral. A third way: Use an identity to write $\sin x \cos x = (1/2)\sin 2x$.

   (e) One way: let $u = 9+x^2$, and make a substitution. A second approach is to make the trigonometric substitution $x = 3\tan\theta$.

2. (a) $\ln|\sec x + \tan x| + C$

   (b)

$$
\begin{aligned}
\int \sec x\ dx &= \int \frac{1}{\cos x}\ dx \\
&= \int \frac{\cos x}{\cos^2 x}\ dx \\
&= \int \frac{\cos x}{1-\sin^2 x}\ dx \\
&= \int \frac{1}{1-u^2}\ du \\
&= \frac{1}{2}\int \left(\frac{1}{1+u} + \frac{1}{1-u}\right)\ du \\
&= \frac{1}{2}\left(\ln|1+u| - \ln|1-u|\right) + C \\
&= \frac{1}{2}\left(\ln|1+\sin x| - \ln|1-\sin x|\right) + C.
\end{aligned}
$$

(c)

$$\begin{aligned}
\int \sec x \, dx &= \int \frac{1}{\cos x} \, dx \\
&= \int \frac{1}{\sin\left(x + \frac{\pi}{2}\right)} \, dx \\
&= \int \frac{1}{\sin u} \, du \\
&= \int \frac{1}{2\sin\left(\frac{u}{2}\right)\cos\left(\frac{u}{2}\right)} \, du \\
&= \int \frac{1}{\sin w \cos w} \, dw \\
&= \int \frac{\sec^2 w}{\sin w \cos w \sec w} \, dw \\
&= \int \frac{\sec^2 w}{\tan w} \, dw \\
&= \int \frac{1}{U} \, dU \\
&= \ln |U| + C \\
&= \ln |\tan w| + C \\
&= \ln \left|\tan \frac{u}{2}\right| + C \\
&= \ln \left|\tan \frac{x + \pi/2}{2}\right| + C.
\end{aligned}$$

3. Evaluate the following integrals:

(a) $x(\ln x)^2 - 2x\ln x + 2x + C$ (b) $\frac{2}{3}\sec^{3/2} x + C$ (c) $\frac{\sqrt{5}x}{\sqrt{5 - x^2}} + C$

(d) $2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$ (e) $x\ln(1+x^2) - 2x + 2\tan^{-1} x + C$ (f) $-\frac{\sqrt{4 + e^{2x}}}{4e^x} + C$

(g) $\frac{3}{2}\sin^{-1}(x-3) + \frac{3}{2}(x-3)\sqrt{6x - x^2 - 8} - \frac{1}{3}(6x - x^2 - 8)^{3/2} + C$

4. (a) $3\pi/32$ (Hint: Let $x^{1/3} = \sin\theta$, i.e., $x = \sin^3\theta$, and substitute.)