Merit Worksheet #6, 1/28/08

Partial fractions

1. Find the integrals (a) $\int \frac{2x+2}{x^2-2x}\,dx$ and (b) $\int \frac{x}{(x-1)^2}\,dx$.

2. The process of breaking a rational function into its smallest "pieces" is called finding the *partial fraction decomposition*. From what you've seen so far, write down <u>on the board</u> what you'd you say are the steps of finding a partial fraction decomposition.

3. Set up the *form* of the partial fractions for each rational function (but do not finish finding the actual partial fraction decomposition).

   Example:
   $$\frac{x^2}{(x-1)^2(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

   (a) $\frac{2x+2}{x^2-2x}$ (b) $\frac{x^2-4x+1}{(x+1)(x-2)^2}$ (c) $\frac{5x^3-5x^2+x-2}{x^2(x^2+1)}$ (d) $\frac{1}{(x^4-1)^3}$

4. Suppose you're finding a partial fraction decomposition, and you've written
   $$\frac{7x-6}{(x+1)(2-3x)} = \frac{A}{x+1} + \frac{B}{2-3x}.$$
   You multiply both sides of the equation by $(x+1)(2-3x)$ (the denominator of the left-hand side) to get
   $$7x-6 = A(2-3x) + B(x+1).$$
   You wish to substitute into this equation values of x that will make the resulting equation very simple. What values of x would do that? What are the equations (with A and B) you would get?

   Example: Plugging in $x=5$, you get
   $$29 = -13A + 6B.$$
   You can get simpler equations if you choose just the right x's. Which x's are they?

5. Find the partial fraction decomposition of the following rational functions:

   (a) $\frac{5x-2}{x^2-4}$

   (b) $\frac{3}{x^3-1}$ (Hint: the denominator can be factored!)

   (c) $\frac{x^2-4x+1}{(x+1)(x-2)^2}$

   (d) $\frac{x^5+x^4+6x^3-4x^2+x-2}{x^2(x^2+1)}$

6. Find the following integrals:

(a) $\displaystyle\int \frac{5x-2}{x^2-4}\,dx$ (b) $\displaystyle\int \frac{x^2-4x+1}{(x+1)(x-2)^2}\,dx$ (c) $\displaystyle\int \frac{x^5+6x^3-4x^2+x-2}{x^2(x^2+1)}\,dx$

7. In order to find the integral of the function in part (b) of Problem 3, you'd have to be able to find the integral

$$\int \frac{dx}{x^2+x+1}.$$

Find this integral.

8. Consider the integral

$$\int \frac{4x}{x^4+4}\,dx.$$

(a) Use partial fractions to solve the integral. Hint: It may help to write

$$x^4+4=(x^4+4x^2+4)-4x^2.$$

(b) Wondering if there's an easier way to find this integral than the way you did it in part (a), you look to see if there's a u-substitution that would make the integral simpler. Can you find one? If so, do you get the same answer in this way that you did in part (a)?

9. Glancing in the back of some textbook, you find the following formula:

$$\int \frac{u^2}{a^2-u^2}\,du = -u+\frac{a}{2}\ln\left|\frac{a+u}{a-u}\right|+C.$$

Derive this formula.

10. Suppose you needed to find a whole bunch of derivatives of the function

$$y=\frac{1}{x^2-1}$$

at the point $x=a$.[1]

a) Using the chain and product rules as necessary, find the first three derivatives of $y=(x^2-1)^{-1}$.
b) Find the partial fraction decomposition of $\frac{1}{x^2-1}$.
c) Now using your answer to part (b), can you find a formula for the nth derivative of y, no matter what n is?

Reading assignment for Wednesday, 1/30: Reread the entire section, making sure you can work each of the examples. Write up exercise 17, showing all your steps, and bring it to class to turn in.

Quote of the day:
"A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction."
—Count Lev Nikolgevich Tolstoy (1828-1920)

[1] When we learn about Taylor series in Chapter 8, you'll see why we might need to do this.