Merit Worksheet #12, 2/15/08

The definition of convergence

We're going to look at the definition of convergence of a sequence again. Let $\{a_n\}_{n=1}^{\infty}$ denote a sequence (remember, a sequence is just a list of numbers). Sometimes the sequence terms get closer and closer to some number L. If that's the case, we say that the sequence is convergent and has a limit L, and we write $\lim_{n\to\infty} a_n = L$. If a sequence is not convergent, it is called divergent.

Now let's review the formal definition of convergence. A sequence $\{a_n\}_{n=1}^{\infty}$ is convergent if there exists a number L such that the numbers a_n get closer and closer to L as n gets larger. We have to make sure that the a_n's really get close to L. One thing we want to avoid is having a_{100} and the terms right after it close to L, while $x_{10,000}$ is far away from L. When we say that $\{a_n\}_{n=1}^{\infty}$ converges to L, we mean that as n gets large, regardless of how close you want a_n to be to L, if you go far enough you will get there. If we want the sequence terms to be match L to 100 decimal places (this happens when $|a_n - L| < 10^{-100}$), then we should have some way of showing that if n is "big enough," each a_n will be that close to L. In other words, let's suppose that ϵ is a very small number denoting the difference allowed between a_n and L (the "error"). Then there exists some point N such that for every term a_n after the Nth term, we have

$$L - \epsilon < a_n < L + \epsilon, \qquad \text{which says the same thing as} \qquad |a_n - L| < \epsilon.$$

To say that again, the integer N tells you how far you have to go to get a_n closer to L than ϵ. (So of course N will depend on ϵ.)

Putting it all into a formal definition: The sequence $\{a_n\}_{n=1}^{\infty}$ converges to the number L if, for every positive number ϵ, there exists some number N such that for every n which is greater than N, $|a_n - L| < \epsilon$.

After reading the preceding paragraphs, answer the following questions.

1. The sequence
$$a_n = \frac{1}{n^2}$$
converges to zero.

   (a) Say I want to list enough terms of the sequence to show that eventually the first three decimal places of the terms all become zero. (This is the same as showing that eventually $|a_n - 0| < 1/1000$.) How big does n have to be to guarantee that this is true?

   (b) Say I want to know how many terms of the sequence I have to go through before the sequence terms all have zero in their first 10 decimal places, i.e., $|a_n| < 10^{-10}$. How big does n have to be?

   (c) Say I want to guarantee that the terms a_n are each within ϵ of 0. How big does n have to be?

2. (Extra practice—Come back to this one when the rest of the worksheet's done.) The sequence
$$a_n = \frac{3^{n+1} - 1}{2 \cdot 3^n}$$
converges to 3/2 (try plugging a few terms into your calculator to convince yourself of this).

   (a) Say I want to list enough terms of the sequence to show that eventually the first three digits of the terms are 1.50. (This is the same as showing that eventually $|a_n - 3/2| < 1/100$.) How big does n have to be to guarantee that this is true?

   (b) Say I want to know how many terms of the sequence I have to go through before the sequence terms all agree with 3/2 to 10 decimal places, i.e., $|a_n - 3/2| < 10^{-10}$. How big does n have to be?

   (c) Say I want to guarantee that the term a_n is within ϵ of 3/2. How big does n have to be?

3. (a) Can a sequence $\{a_n\}_{n=1}^{\infty}$ converge to two different numbers?
   (b) Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence obtained by interspersing the members of the two sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$, each of which converges. Does it follow that $\{a_n\}_{n=1}^{\infty}$ is convergent? If so, explain why; if not, show by example why not.
   (c) Suppose infinitely many of the terms of the convergent sequence $\{a_n\}_{n=1}^{\infty}$ belong to the interval (b, c). Must the limit of the sequence belong to the interval (b, c)?

The Squeeze Theorem

4. (a) To what do the sequences $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and $\left\{-\frac{1}{n}\right\}_{n=1}^{\infty}$ converge?
   (b) Sketch, on the same set of axes, graphs of the sequences $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, $\left\{-\frac{1}{n}\right\}_{n=1}^{\infty}$, and $\left\{\frac{1}{2n}\right\}_{n=1}^{\infty}$.
   (c) Give a reasonably convincing argument, referring to your picture, that, if you know for sure that $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and $\left\{-\frac{1}{n}\right\}_{n=1}^{\infty}$ both converge to 0, then $\left\{\frac{1}{2n}\right\}$ also converges to 0.
   (d) Refer to the Squeeze Theorem on page 617 of your text. Is your answer to part (c) essentially the same thing as what the Squeeze Theorem says? In this example, what would a_n, b_n, c_n, and L all be?

5. Using the Squeeze Theorem (rather than the definition of convergence from Problems 1 and 2), prove that the sequence $\left\{\frac{\cos(n)}{n!}\right\}_{n=1}^{\infty}$ converges to 0.

Monotonic and bounded sequences

6. (a) What does the word "monotonous" mean in everyday English?
   (b) What does it mean for a sequence to be increasing? Decreasing? Monotonic? (See page 619 of your text.) Can you see a connection between the definition of "monotonic" and your answer to part (a)?
   (c) Which of the following sequences $\{a_n\}_{n=1}^{\infty}$ are monotonic?

   (i) $a_n = \dfrac{2n}{5n-3}$ (ii) $a_n = \dfrac{1+(-1)^n}{\sqrt{n}}$ (iii) $a_n = \tan n$ (iv) $1-(2/3)^n$

   (d) Does a monotonic sequence have to be convergent? If so, explain why; if not, give an example of a divergent monotonic sequence.
   (e) Does a convergent sequence have to be monotonic? If so, explain why; if not, give an example of a non-monotonic convergent sequence.

7. (a) What does it mean for a sequence to be bounded?
   (b) Does a bounded sequence have to be convergent? If so, explain why; if not, give an example of a divergent bounded sequence.
   (c) Does a convergent sequence have to be bounded? If so, explain why; if not, give an example of an unbounded convergent sequence.
   (d) Which of the following sequences are bounded?

   (i) $a_n = \dfrac{2n}{5n-3}$ (ii) $a_n = [1+(-1)^n]n$ (iii) $a_n = \sqrt{\dfrac{2+\cos n}{n}}$ (iv) $1-(3/2)^n$

(e) Does a sequence which is both bounded and monotonic have to be convergent?

8. Find a sequence of rational numbers that is guaranteed to converge to π.

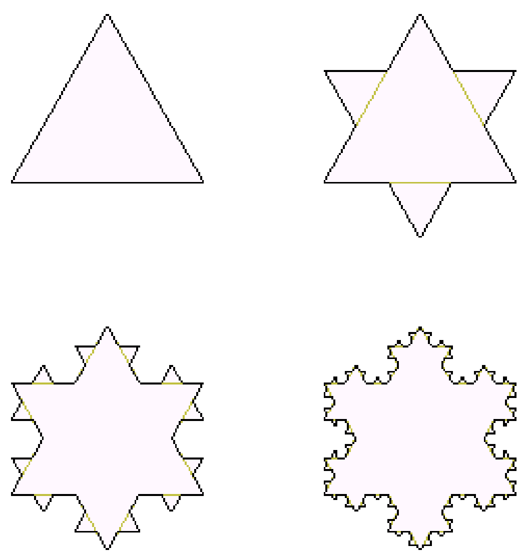
A review problem

From now on I'll include a review problem or two at the end of each worksheet. The idea is that if you work these problems diligently, you'll be a lot more prepared when the final rolls around than you otherwise would be. A complete solution will appear online in the answer key to the worksheet. Please let me know what types of problems you could use some extra practice in. Here's the first one:

Evaluate the integral $\int x^2 \tan^{-1} x \, dx$.

Preparation assignment for Monday, 2/18: On Monday we will begin our study of Section 8.2 in the text. Please read pages 626 through the end of Example 2.3 on page 629, and the two sentences immediately following the example. Then skim the rest of the section. For your preparation assignment, please turn in Exercise 26.

Mathematical-oddity-about-sequences of the day:
Have you seen the Koch snowflake before? It's a *fractal* (a shape which is detailed—and often looks exactly the same—no matter how closely you zoom in on it) formed by starting with an equilateral triangle and then successively taking straight line segments and putting a "kink" in them. The first couple of iterations appear below; the snowflake



It's not surprising, perhaps, that the length of the snowflake is infinite. You can see that as you perform more iterations, adding more detail to the snowflake, the perimeter increases (you might say the perimeter is a monotone increasing sequence!). But here's what's remarkable: the same thing happens in unexpected places. The following is an excerpt from the Wikipedia article on the mathematician Lewis Fry Richardson (1881-1953):

“While studying the causes of war between two countries, Richardson decided to search for a relation between the probability of two countries going to war and the length of their common border. While collecting data, he realized that there was considerable variation in the various gazetted lengths of international borders. For example, that between Spain and Portugal was variously quoted as 987 or 1214 km while that between The Netherlands and Belgium as 380 or 449 km.

“As part of his research, Richardson investigated how the measured length of a border changes as the unit of measurement is changed. He published empirical statistics which led to a conjectured relationship. This research was quoted by mathematician Benot Mandelbrot in his 1967 paper ‘How Long Is the Coast of Britain?’

“Suppose the coast of Britain is measured using a 200 km ruler, specifying that both ends of the ruler must touch the coast. Now cut the ruler in half and repeat the measurement, then repeat again. [See the picture below.]

“Notice that the smaller the ruler, the bigger the result. It might be supposed that these values would converge to a finite number representing the ‘true’ length of the coastline. However, Richardson demonstrated that the measured length of coastlines and other natural features appears to *increase without limit* as the unit of measurement is made smaller. Today this is known as the Richardson effect.

“Note that Richardson’s results do *not* mean that the coastline of Britain is actually infinitely long. This would require the ability to measure with infinitesimally small rulers, something which quantum physics says cannot be done, as there is a lower limit to the smallness of a measurement, the Planck length. What Richardson’s results do show is that natural geographic features, when considered over a wide range of scales, do not behave in the same way as the objects of Euclidean geometry.

“At the time, Richardson’s research was ignored by the scientific community. Today, it is seen as one element in the birth of the modern study of fractals.”

(Italics added; all images from wikipedia.org)

Cool, huh?