Selected answers for Merit Worksheet #13

1. A series is a sum (of the terms in a sequence). Three examples might include $\sum_{k=1}^{\infty} \frac{1}{k}$, $\sum_{k=1}^{\infty} k$, and $\sum_{k=1}^{\infty} 2^k$. A series differs from a sequence in that a series is a *sum* of a list of numbers; a sequence is just a list of numbers.

2. Jason and Marcus are listing elements of a *sequence*, not a series.

3. A partial sum of a series is a sum of a finite number of terms of the series. For the examples above, the first four partial sums are as follows:

   $\sum_{k=1}^{\infty} \frac{1}{k}$: $S_1 = \frac{1}{1}$, $S_2 = \frac{1}{1} + \frac{1}{2}$, $S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3}$, and $S_4 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$.

   $\sum_{k=1}^{\infty} k$: $S_1 = 1$, $S_2 = 1 + 2 = 3$, $S_3 = 1 + 2 + 3 = 6$, and $S_4 = 1 + 2 + 3 + 4 = 10$.

   $\sum_{k=1}^{\infty} 2^k$: $S_1 = 2$, $S_2 = 2^1 + 2^2 = 6$, $S_3 = 2^1 + 2^2 + 2^3 = 14$, and $S_4 = 2^1 + 2^2 + 2^3 + 2^4 = 30$.

4. See pages 627 through 629 of the text.

5. The table shows the partial sums of the series whose terms come from the sequence $\{a_n\}_{n=1}^{\infty}$. The partial sums appear to be approaching a limit of approximately 1.71828, so the series seems to converge.

6. (a) The sequence has limit 0.

   (b) The partial sums are as follows:

   $$S_1 = \frac{1}{4}, \qquad S_2 = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}, \qquad S_3 = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16}, \qquad S_4 = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{15}{32},$$

   $$S_5 = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} = \frac{31}{64}.$$

   (c) The series seems to converge to 1/2.

   (d) Part (a) asks for the limit of the *sequence*; part (c) asks for the limit of the *partial sums*.

7. (a) The sequence has limit 1.

   (b) The partial sums are as follows:

   $$S_1 = \frac{1}{2} = 0.5, \qquad S_2 = \frac{1}{2} + \frac{2}{3} = \frac{7}{6} = 1.16666..., \qquad S_3 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12} = 1.916666...,$$

   $$S_4 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} = \frac{163}{60} = 2.716666..., \qquad S_5 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} = \frac{71}{20} = 3.55.$$

   (c) The partial sums seem to be growing without approaching any particular number.

   (d) The series does not seem to converge (even though the sequence of individual terms does converge).

9. $\sum_{k=0}^{\infty} \frac{1}{3^k}$, $\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}}$, $\sum_{k=1}^{\infty} 1$, $\sum_{n=2}^{\infty} \frac{10}{3 \cdot 7^n}$, and $\sum_{n=0}^{\infty} \pi^n$ are geometric series. The others are not.

10. We find

$$\sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{1}{1-(1/3)} = \frac{3}{2};$$

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{5}\left(\frac{2^2}{5}\right)^n = \frac{4}{5};$$

$$\sum_{k=1}^{\infty} 1 \quad \text{diverges;}$$

$$\sum_{n=2}^{\infty} \frac{10}{3\cdot 7^n} = \sum_{n=2}^{\infty} \frac{10}{3}\left(\frac{1}{7}\right)^n = \frac{5}{63};$$

$$\sum_{n=0}^{\infty} \pi^n \quad \text{diverges.}$$

11. True—just check whether or not $|r| < 1$.

12. The series is called a telescoping sum because of the "collapsing" that happens in the partial sums, similar to the way an old telescope or spyglass collapses down.

    (a) The nth partial sum is $S_n = \ln(n+1)$, which approaches ∞ as n approaches ∞, so the series diverges.

    (b) The nth partial sum of the series is

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{k(k+2)} \\ &= \sum_{k=1}^{n} \left[\frac{1/2}{k} - \frac{1/2}{k+2}\right] \\ &= \left(\frac{1/2}{1} - \frac{1/2}{3}\right) + \left(\frac{1/2}{2} - \frac{1/2}{4}\right) + \left(\frac{1/2}{3} - \frac{1/2}{5}\right) + \cdots + \left(\frac{1/2}{n-1} - \frac{1/2}{n+1}\right) + \left(\frac{1/2}{n} - \frac{1/2}{n+2}\right) \\ &= \frac{1/2}{1} + \frac{1/2}{2} + -\frac{1/2}{n+1} - \frac{1/2}{n+2}. \qquad \text{(the only terns which don't cancel out)} \end{aligned}$$

    As n approaches infinity, the partial sums approach $(1/2)/1 + (1/2)/2$, so the series converges to $3/4$.

13. The harmonic series is the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$. The harmonic series diverges.

Review problems

A. Find $\int \frac{1}{x\sqrt{1+x^2}}\, dx$. *B. Find* $\int_0^{\infty} \frac{x}{1+x^2}\, dx$.

A. We make the trigonometric substitution $x = \tan\theta$. Then $dx = \sec^2\theta\, d\theta$. The integral becomes

$$\int \frac{1}{x\sqrt{1+x^2}}\, dx = \int \frac{\sec^2\theta}{\tan\theta\sec\theta}\, d\theta = \int \csc\theta\, d\theta = -\ln|\csc\theta + \cot\theta| + C.$$

In order to simplify these trig expressions, we draw and label a triangle. Since we made the substitution $\tan\theta = x$, we mark an angle as θ, label its opposite side as x, and label its adjacent side as 1. This makes the hypotenuse equal to $\sqrt{1+x^2}$, and from the triangle we substitute x's back in, finding

$$\int \frac{1}{x\sqrt{1+x^2}}\,dx = -\ln|\csc\theta + \cot\theta| + C = \boxed{-\ln\left|\frac{\sqrt{1+x^2}}{x} + \frac{1}{x}\right| + C.}$$

B. We use the limit definition of the improper integral, as well as the substitution $u = 1 + x^2$, to evaluate the integral:

$$\begin{aligned}\int_0^\infty \frac{x}{1+x^2}\,dx &= \lim_{b\to\infty} \int_0^b \frac{x}{1+x^2}\,dx \\ &= \lim_{b\to\infty} \left[\frac{1}{2}\ln|1+x^2|\right]_0^b \\ &= \lim_{b\to\infty} \frac{1}{2}\ln|1+b^2| \\ &= \infty.\end{aligned}$$

Therefore, the integral diverges.