Selected answers to Merit Worksheet #16

1.

> A series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p-**series**.
>
> A p-series written this way will converge **exactly when** $p > 1$.

2. (a) Converges (compare to $\sum 1/n^2$).
   (b) Diverges (compare to the harmonic series).

3. (a) Diverges (compare to the harmonic series).
   (b) Converges (compare to $\sum 1/n^2$).

4. There are many possible ways to arrive at these answers. The method in parentheses is one method.
   (a) Diverges (Limit comparison test with the harmonic series).
   (b) Diverges (Comparison test with the harmonic series).
   (c) Diverges (Comparison test with the harmonic series).
   (d) Converges (Limit comparison test with $\sum 1/k^{3/2}$).
   (e) Diverges (Limit comparison test $\sum (3/e)^k$).
   (f) Converges (Comparison test with $\sum (1/2)^k$).
   (g) Diverges (Limit comparison test $\sum 1/\sqrt{k}$).
   (h) Converges (Comparison test with $\sum 1/k^2$).
   (i) Diverges (Integral test).
   (j) Diverges (Limit comparison test with the harmonic series)
   (k) Converges (Comparison test with $\sum (2/e)^n$).

5. The series in (a) converges by comparison with the series $\sum (1/e)^k$. The integral in (b) converges by comparison with the integral $\int_1^{\infty} e^{-x}\,dx$. Both the comparison test for series and the comparison test for improper integrals compare the given series term or integrand with a smaller or larger term or integrand. Of course, if the improper integral is doable, then evaluating it well tell us right away whether or not the corresponding series converges. Thus, evaluating improper integrals and infinite series are often similar problems with similar approaches (though not always, as we'll see next week).

6. We'll assume that each a_k is positive. Since $\sum a_k$ converges, we know that $\lim_{k\to\infty} a_k = 0$ (otherwise, the kth-term test for divergence would tell us that the series *doesn't* converge). Since the limit is 0, eventually we'll have $a_k < 1$. When this happens, $a_k^2 < a_k$. Then the comparison test tells us that since $\sum a_k$ converges, we know that the series $\sum a_k^2$ converges as well.

7. Take $a_n = 1/n$ and $b_n = 1/n^2$, for example. The limit comparison test says nothing about when the limit equals 0.

Review problem

(A) The series $\sum_{n=1}^{\infty} \sin^n 1$ is a geometric series. Since $|\sin 1| < 1$, the series converges to

$$\frac{\sin 1}{1 - \sin 1}.$$

(B) Since $\lim_{n \to \infty} \frac{n^2 - 1}{3n^2 + 1} = \frac{1}{3}$, which does not equal zero, the kth-term test for divergence tells us that the series diverges.