Selected answers to Merit Worksheet #20

1. A sequence is a list of numbers. For the definition of convergence, read the paragraphs immediately preceding Definition 1.1 on page 613 (Section 8.1) of your text.

2. (a) Converges to $-4/3$.
   (b) Diverges.
   (c) Converges to 2.
   (d) Converges to 1. (Rewrite the limit's function as $\sin(1/n)/(1/n)$, and then use l'Hospital's rule.)

3. $\sum_{k=0}^{\infty} \frac{1}{3^k}$, $\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}}$, $\sum_{k=1}^{\infty} 1$, $\sum_{n=2}^{\infty} \frac{10}{3 \cdot 7^n}$, and $\sum_{n=0}^{\infty} \pi^n$ are geometric series. The others are not.

4. We find

$$\sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{1}{1-(1/3)} = \frac{3}{2};$$

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{5}\left(\frac{2^2}{5}\right)^n = \frac{4}{5};$$

$$\sum_{k=1}^{\infty} 1 \quad \text{diverges;}$$

$$\sum_{n=2}^{\infty} \frac{10}{3 \cdot 7^n} = \sum_{n=2}^{\infty} \frac{10}{3}\left(\frac{1}{7}\right)^n = \frac{5}{63};$$

$$\sum_{n=0}^{\infty} \pi^n \quad \text{diverges.}$$

5. (a) False; the kth-term test tells us that the series actually diverges.
   (b) True; since the limit is not zero, the kth-term test tells us that the series diverges.
   (c) False; since the limit of the terms is zero, the kth-term test *does not apply*, and does *not* tell us that the series converges. (In fact, the series diverges, which you can see by using the integral test.)
   (d) False; the series does converge, but not just because $\lim_{k\to\infty} 1/2^k = 0$; and anyway, the series obviously has a sum greater than the claimed value of 0.
   (e) True; the individual terms must approach 0.
   (f) False; look at the harmonic series, for example.

6. Since

$$\frac{1}{k(k+2)} = \frac{1/2}{k} - \frac{1/2}{k+2},$$

which we find by finding the partial fraction decomposition of $1/(k(k+2))$, we see that the nth partial sum of the series is

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \frac{1}{k(k+2)} \\
&= \sum_{k=1}^{n} \left[\frac{1/2}{k} - \frac{1/2}{k+2}\right] \\
&= \left(\frac{1/2}{1} - \frac{1/2}{3}\right) + \left(\frac{1/2}{2} - \frac{1/2}{4}\right) + \left(\frac{1/2}{3} - \frac{1/2}{5}\right) + \cdots + \left(\frac{1/2}{n-1} - \frac{1/2}{n+1}\right) + \left(\frac{1/2}{n} - \frac{1/2}{n+2}\right) \\
&= \frac{1/2}{1} + \frac{1/2}{2} + -\frac{1/2}{n+1} - \frac{1/2}{n+2}. \qquad \text{(the only terns which don't cancel out)}
\end{aligned}$$

As n approaches infinity, the partial sums approach $(1/2)/1 + (1/2)/2$, so

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{3}{4}.$$

For another example in telescoping series, see Example 2.3 on page 628 of your text.

7. A p-series is a series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$. It converges when when $p > 1$ and diverges when $p \leq 1$. You can prove the rule by using the integral test. (You can also prove the divergence part of it by using the comparison test with the harmonic series.)

8. For the integral test to be applicable, the series term needs to be a function of k which is *continuous* on the real numbers, always *decreasing*, always *positive.* The integral test will be useful when the series term represents a function whose integral is easily found, such as when the series term has an easily-found antiderivative.

9. (a) Converges
   (b) Diverges
   (c) Diverges

10. False; the integral and the series are *similar* in size, but they are not *equal* in size.

11. (a) Converges (comparison with $\sum 1/k^2$).
    (b) Diverges (comparison with the harmonic series).

12. (a) Diverges (limit comparison test with the harmonic series).
    (b) Converges (limit comparison test with $\sum 1/k^2$).

13. The series satisfies the conditions of the alternating series test, so it converges.

14. It converges conditionally, since it converges, but the series

$$\sum_{k=1}^{\infty} \left|\frac{(-1)^k k}{3k^2+2}\right| = \sum_{k=1}^{\infty} \frac{k}{3k^2+2}$$

diverges (limit comparison test with the harmonic series).

15. (a) Converges (the limiting ratio is 1/2).
    (b) Converges (the limiting ratio is 0).

16. In both cases the limiting ratio is 1, so the ratio test doesn't tell you anything about whether or not the series converges.

17. (a) Diverges (the limiting root is 2).
    (b) Converges (the limiting root is $1/e$).

18. The limiting root is 1.

19. Here are each of the series, along with suggestions of appropriate tests to try.

$\displaystyle\sum_{k=1}^{\infty} \frac{\sin^2 k}{3^k}$ converges, by comparison with $\sum (1/3)^k$.

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k+\sqrt{k}}$ diverges, by limit comparison with the harmonic series.

$\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{2}{k^2}$ converges, by the alternating series test.

$\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{k+1}$ diverges, by the kth-term test for divergence.

$\displaystyle\sum_{k=1}^{\infty} \frac{1+\sqrt[3]{k}}{k}$ diverges, by comparison with the harmonic series, or by limit comparison with $\sum 1/k^{2/3}$.

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k\sqrt{k}}$ converges, by comparison with $\sum (1/2)^k$.

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^k}$ converges, by the root test, or the ratio test, or comparison with $\sum 1/k^2$, or comparison with $\sum (1/2)^k$.

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{[3+(-1)^k]^k}$ converges, by comparison with $\sum (1/2)^k$.

$\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{2^k}$ converges, by the alternating series test.

$\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{4^k}{k!}$ converges, by the alternating series test.

$\displaystyle\sum_{k=1}^{\infty} k^3 \left(\frac{3}{4}\right)^k$ converges, by the root test.

$\displaystyle\sum_{k=1}^{\infty} \frac{k}{2k+1}$ diverges, by the kth-term test for divergence.

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2-3}$ converges, by limit comparison with $\sum 1/k^2$.

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{2k^3-k}}$ converges, by limit comparison with $\sum 1/k^{3/2}$.

$\displaystyle\sum_{k=1}^{\infty} \frac{3^k+1}{5e^k+k}$ diverges, by limit comparison with $\sum (3/e)^k$.

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{4k-1}$ diverges, by limit comparison with the harmonic series, or by comparison with $\sum 1/(4k)$.

$\displaystyle\sum_{k=1}^{\infty} \frac{5+\sqrt{k}}{1+k}$ diverges, by limit comparison with $\sum 1/k^{1/2}$.

$\displaystyle\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$ converges, by the ratio test (the limiting ratio is 1/2).

$\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{k!}$ converges, by the ratio test.

$\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ converges, by the ratio test (the limiting ratio is 1/4).

$\displaystyle\sum_{k=1}^{\infty} \frac{k}{(k+1)(k+3)(k+5)}$ converges, by limit comparison with $\sum 1/k^2$.

$\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{k!}$ diverges, by the ratio test (the limiting ratio is e).

20. The series will converge exactly if the degree m of the denominator is more than 1 larger than the degree ℓ of the numerator. We find this applying the limit comparison test with $\sum 1/k^{m-\ell}$.

21. See, for example, the number lines drawn in Figures 8.31 and 8.32 on pages 649 and 650 of your text.

22. The farthest the partial sum $\displaystyle\sum_{k=1}^{n} (-1)^{k+1} a_k$ (the nth partial sum, also known as S_n) can be away from the total series value is a_{n+1}. This is the meaning of the inequality

$$|S_n - L| \leq a_{n+1}$$

which we discussed in class.

23. We know that $|S_n - L| \leq a_{n+1}$. We require that $|S_n - L| \leq 1/100$. This will automatically be true if $a_{n+1} \leq 1/100$, i.e., if

$$\frac{1}{(n+1)^2} \leq \frac{1}{100},$$
$$(n+1)^2 \geq 100,$$
$$n+1 \geq 10,$$
$$\boxed{n \geq 9.}$$

So if we want the partial sum to be within 1/100 of the sum of the series, we should include at least 9 terms in our partial sum.