Selected answers to Merit Worksheet #21

1. The series in (a) and (c) converge; those in (b) and (d) diverge.
2. The series may converge or diverge, based upon what x is. If $|x| < 1$, the series converges, while if $|x| \geq 1$, the series diverges.
3. The series in (a) converges by the alternating series test. The series in (b) converges, because each term equals 0. The series in (c) diverges, by the kth-term test for divergence. The series in (d) diverges, because it equals the harmonic series.
4. Here's a full solution: We apply the ratio test to the series, and find that

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}(x+1)^{k+1}}{k+1}\cdot\frac{k}{(-1)^k(x+1)^k}\right| \\ &= \lim_{k\to\infty}\left|\frac{(x+1)k}{k+1}\right| \\ &= |x+1|. \qquad (1)\end{aligned}$$

Note that *it is very important to put the absolute value signs around the fraction;* the ratio test *requires* that they be there. Now the ratio test says that if the limit we found above is less than 1, then the series converges, and if it's greater than 1, then the series diverges.

So first, of all, when is the limit in (1) less than 1? There's a trick for working with inequalities and absolute values. Whenever you have the inequality $|A| < c$, you can take the absolute value sign off by replacing this first inequality by another, namely

$$-c < A < c.$$

The two inequalities mean exactly the same thing and have exactly the same solutions. That means that

$$|x+1| < 1 \qquad \text{means the same thing as} \qquad -1 < x+1 < 1.$$

Solving that second inequality gives us $-2 < x < 0$. Thus, the limit in (1) from the ratio test is less than 1, and hence the series diverges, when x is between -2 and 0.

When is the limit in (1) greater than 1? It shouldn't be surprising that this happens outside the interval we found above, i.e., when $x < -2$ or when $x > 0$. For these values of x, the series diverges.

Now remember that the ratio test tells us *nothing* about the convergence of the series when the limit we find equals 1. So when does the limit in (1) equal 1? To solve $|x+1| = 1$, we recall that the absolute value of a quantity either gives us back the same thing, or it changes the sign. Therefore, $|x+1|$ is either the same as $x+1$, or $|x+1| = -(x+1)$. So to solve $|x+1| = 1$, we solve the two equations

$$x+1 = 1 \qquad \text{and} \qquad -(x+1) = 1,$$

which give us the two solutions $x = 0$ and $x = -2$. The ratio test doesn't tell us whether or not the series converges when x equals either of these two numbers, so we have to use some other test to determine the convergence or divergence.

Let's look first at what happens when $x = -2$. Actually, when you plug $x = -2$ into the series, you get

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}(-2+1)^k = \sum_{k=1}^{\infty}\frac{[(-1)^k]^2}{k} = \sum_{k=1}^{\infty}\frac{1}{k},$$

which is the harmonic series, which we know diverges. Thus, -2 is *not* a value of x for which the series converges.

How about when $x = 0$? Plugging this into the series yields

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}(0+1)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} 1^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k},$$

which converges by the alternating series test. Thus 0 *is* a value of x for which the series converges. Putting it all together, the set of x's for which the series converges is the interval $(-2, 0]$.

5. We use the ratio test:

$$\begin{aligned} \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k\to\infty} \left| \frac{(k+1)!(x-2)^{k+1}}{k!(x-2)^k} \right| \\ &= \lim_{k\to\infty} |(k+1)(x-2)|\,. \end{aligned} \tag{2}$$

Now if x equals 2, then the limit in (2) equals 0, which is less than 1, so the series converges. However, if x is anything other than 2, the limit in (2) does not exist (it's infinite), so the ratio test tells us that the series diverges. Therefore, the collection of x's for which the series converges is just the single x-value 2 (which can be written, if you like, as the interval $[2, 2]$).

7. For Problem 2, the interval of convergence is $(-1, 1)$, and the radius of convergence is 1. For Problem 4, the interval of convergence was $(-2, 0]$, and the radius of convergence is 1. For Problem 5, the interval of convergence is $[2, 2]$, and the radius of convergence is 0.

8. (a) True: just as in Problem 4, the ratio test always gives you an interval of x's for which the series converges. It does not tell you whether or not the series converges when x equals one of the endpoints of this interval, so it doesn't tell you everything there is to know about the *interval* of convergence, but it does tell you the *length* of the interval, which is all you need to know to find the *radius* of convergence.

   (b) Because at these points, the limit you find in the ratio test equals 1, and the ratio test is inconclusive.

9. (a) Interval of convergence: $(-3, 9)$; radius of convergence: 6.

   (b) Interval of convergence: $[-1, 1]$; radius of convergence: 1.

   (c) This one's interesting. If you use the ratio test, you get

$$\begin{aligned} \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right| \\ &= \lim_{k\to\infty} \frac{|x|}{k+1}. \end{aligned}$$

   Now no matter what x is, this limit always equals 0, which is less than 1, so *no matter what x is, the series will always converge, by the ratio test.* Therefore, the interval of convergence is $(-\infty, \infty)$, and the radius of convergence is infinite.

   (d) The ratio test gives us

$$\begin{aligned} \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k\to\infty} \left| \frac{[\ln(k+1)](x+1)^{k+1}}{k+1} \cdot \frac{k}{(\ln k)(x+1)^k} \right| \\ &= \lim_{k\to\infty} \left| \frac{\ln(k+1)}{\ln k} \cdot \frac{k}{k+1} \cdot (x+1) \right| \\ &= \left(\lim_{k\to\infty} \left| \frac{\ln(k+1)}{\ln k} \right| \right) \left(\lim_{k\to\infty} \left| \frac{k}{k+1} \right| \right) |x+1| \\ &= 1 \cdot 1 \cdot |x+1|. \end{aligned}$$

The inequality $|x+1| < 1$ is the same as saying $-1 < x+1 < 1$, which reduces to $-2 < x < 0$. Therefore, the interval of convergence includes $(-2, 0)$. Checking the endpoints, we see that when $x = -2$ the series converges by the alternating series test, while when $x = 0$ the series diverges by the integral test. Thus the interval of convergence is $[-2, 0)$. The radius of convergence is 1.

10. Since the series is in terms of powers of $(x-1)$, the interval of convergence is centered at $x = 1$. Since $x = 3$ makes the series converge, the radius of convergence is at least 2 (the distance between $x = 1$ and $x = 3$). That means that the series will definitely converge for all x in $(-1, 3)$. Hence the series in (a) converges (there we have $x = 2$); the series in (b) also converges (there $x = 0$); and the series in (c) has $x = -1$. This is exactly 2 units away from the center $x = 1$, and we argued that the radius of convergence was at least 2. However, it's possible that $x = 3$ and $x = -1$ are the endpoints of the interval of convergence, with the series converging at $x = 3$ and diverging at $x = -1$. Therefore, we can't say whether or not the series in part (c) converges.

11. Since the power series has an interval of convergence centered at $x = -2$ (since $x + 2 = x - (-2)$), and converges at $x = 4$, the radius of convergence must be at least 6 (the distance from $x = -2$ to $x = 4$). That means that the interval of convergence must contain $(-8, 4)$. Any x in that interval must make the series converge.

13. By the geometric series formula,

$$\sum_{k=0}^{\infty}(-x)^k = \frac{1}{1-(-x)} = \frac{1}{1+x} \tag{3}$$

whenever $|x| < 1$. Now notice that $\ln(1+x)$ is an antiderivative for $1/(1+x)$. We don't know yet whether this is allowed, but if we take the antiderivative of both sides of (3), we might get

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \sum_{k=0}^{\infty}\int (-1)^k x^k\,dx = \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1}.$$

It seems we have found a power series that equals $\ln(1+x)$ when $|x| < 1$.

Review problem

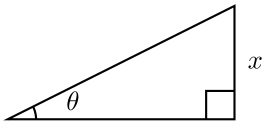
Find the following indefinite integrals:

$$\int \frac{1}{\sqrt{1+x^2}}\,dx \qquad\qquad \int \frac{x+3}{x^2+3x+2}\,dx$$

In the first integral, we notice that the $1 + x^2$ fits the "$a^2 + x^2$" pattern, and that (as well as the square root) suggests that a trig substitution is the way to go. Remember that when the integral has an $a^2 + x^2$, it's usually good to make the substitution $x = a\tan\theta$, so here we'll make the substitution $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, and integrating we get

$$\begin{aligned}\int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,d\theta\\ &= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta\\ &= \int \sec\theta\,d\theta\\ &= \ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

Now we've got to change the θ's back into x's. One way to do that is to draw a triangle illustrating our substitution. Since we made $x = \tan\theta$, our triangle looks like this:



Using the Pythagorean theorem, we find that $\sec\theta = \sqrt{1+x^2}$, so

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln|\sqrt{1+x^2}+x| + C.$$

For the second integral, we use the method of partial fractions (since it's a rational function with a denominator that factors). We factor the denominator and split the fraction up:

$$\frac{x+3}{x^2+3x+2} = \frac{x+3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2};$$

$$x+3 = A(x+2) + B(x+1);$$

Letting x be -2 and then -1 in the equation gives us that $B=-1$ and $A=2$. Then

$$\frac{x+3}{x^2+3x+2} = \frac{2}{x+1} - \frac{1}{x+2},$$

and

$$\int \frac{x+3}{x^2+3x+2}\,dx = \int \left(\frac{2}{x+1} - \frac{1}{x+2}\right)dx = 2\ln|x+1| - \ln|x+2| + C,$$

which is our answer.