Merit Worksheet #22, 3/14/08—Pi Day!

Power series and different values of x

1. (a) Consider the power series $\sum_{k=0}^{\infty} 2^k x^k$. What is the interval of convergence for this series? (Remember to check: does the interval include the endpoints or not?)

   (b) Find the values of the following series:

   (i) $\sum_{k=0}^{\infty} 2^k \left(-\frac{1}{3}\right)^k$ (ii) $\sum_{k=0}^{\infty} 2^k \left(\frac{1}{4}\right)^k$ (iii) $\sum_{k=0}^{\infty} 2^k \left(\frac{\pi}{10}\right)^k$

   (c) If $\sum_{k=0}^{\infty} 2^k x^k$ converges, what does it converge to?

2. Rank the following series in order from smallest sum to largest sum:

   (a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}(-1/3)^k$ (b) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}(1)^k$ (c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}(-0.1)^k$ (d) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}(-3/4)^k$

   (e) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}(1/3)^k$

3. Based on Problem 2, sketch what you *think* the graph of

   $$y = \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}$$

   looks like. Specifically, what x-values correspond to points on the graph? Where is the graph increasing/decreasing?

Power series for special functions

The problems above aim to help you see that on the interval of convergence, a power series defines a *function* of x. We may not be able to write down a nice formula for that function, but sometimes we can.

4. What rational function does the sum $\sum_{k=0}^{\infty} x^k$ equal? For which x are the series and the rational function equal?

5. As you read in preparation for today, if we differentiate (or integrate) each term of a series, it becomes a power series which equals the derivative (or integral) of the function the original function equals. Based on your answer to Problem 4, what functions do the series

   $$\sum_{k=0}^{\infty} kx^{k-1} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

   equal?

6. Use the power series $\sum_{k=0}^{\infty}(-1)^k x^k$ to find power series representations of $\dfrac{1}{(1+x)^2}$, $\dfrac{1}{1+x^2}$, and $\tan^{-1} x$.

7. Use your answer to Problem 5 to find the sum of the following series:

$$\text{(a)} \quad \sum_{k=0}^{\infty} \frac{k}{2^{k-1}} \qquad \text{(b)} \quad \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)3^{k+1}}$$

8. (a) Note that the series you dealt with in Problems 2 and 3, $\sum_{k=1}^{\infty} \dfrac{(-1)^k x^k}{k}$, may be rewritten in a different way:

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k} = -\frac{x^1}{1} + \frac{x^2}{2} - \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty} \frac{(-x)^{k+1}}{k+1}.$$

Based on your answer from Problem 5, what function does this series equal?

(b) Sketch the function you found in part (a). How does it compare to the graph you sketched in Problem 3?

The interval of convergence

You've seen that if you have a power series equal to a function $f(x)$, then to get a power series equal to $f'(x)$, all you need to do is differentiate each term of the original series. The same type of thing happens if you want to find a power series equal to $\int f(x)\,dx$. But what can we say about the intervals of convergence for the series we get?

9. Find the intervals of convergence for the following series:

$$\text{(a)} \quad \sum_{k=0}^{\infty} k x^{k-1} \qquad \text{(b)} \quad \sum_{k=0}^{\infty} x^k \qquad \text{(c)} \quad \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

10. Find the intervals of convergence for the following series:

$$\text{(a)} \quad \sum_{k=1}^{\infty}(-1)^k x^{k-1} \qquad \text{(b)} \quad \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k} \qquad \text{(c)} \quad \sum_{k=1}^{\infty} \frac{(-1)^k x^{k+1}}{k(k+1)}$$

11. What can you say about the *radius* of convergence when you differentiate or integrate a power series (see the top of page 668 of your text). What can you say about how the *interval* of convergence may change when you take the derivative or integral of a power series?

12. (Bonus) Can you explain why the radius of convergence remains unchanged when you differentiate or integrate a power series? (Hint: use the ratio test somehow.)

13. Seeing as it's Pi Day, can you use your answer to Problem 6 to come up with a series that converges to π?

Review problems

A. Find $\displaystyle\int \cos x \sin^3 x\, dx$.

B. Does the integral $\displaystyle\int_0^{\infty} \frac{4}{4+x^2}\, dx$ converge or diverge?

Preparation for next time

After the break, we will spend a week going over Section 8.7 on Taylor series. For Monday, March 24, please read Section 8.7 from the beginning through Example 7.2. Work Exercise 1 from the section, and show all your work.

Quotes of the day

Here are a few about π, from `www.piday.org`:

"Im like π...irrational, but well-rounded." — Anonymous

π to i: "Get real!"
i to π: "Be rational!"

If, in a circle, a line
Hits the center, and runs spine to spine,
And the line's length is D
The circumference will be
D times 3.14159.

A great palindrome: "I prefer pi."