Selected answers to Merit Worksheet #22

1. (a) $(-1/2, 1/2)$
   (b) (i) $3/5$
   (ii) 2
   (iii) $1/[1-(\pi/5)] = 5/(5-\pi)$
   (c) $1/(1-2x)$

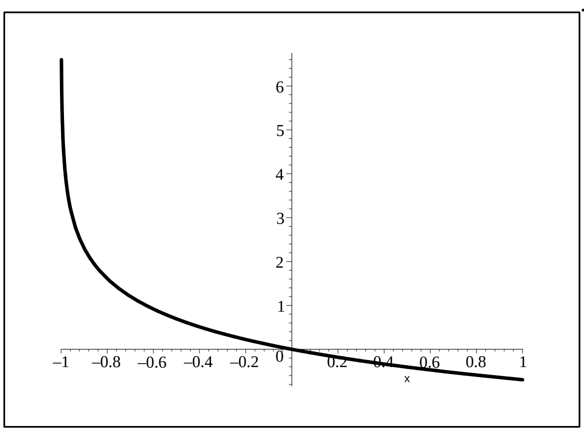
2.
$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}(1)^k < \sum_{k=1}^{\infty}\frac{(-1)^k}{k}(1/3)^k < \sum_{k=1}^{\infty}\frac{(-1)^k}{k}(-0.1)^k < \sum_{k=1}^{\infty}\frac{(-1)^k}{k}(-1/3)^k < \sum_{k=1}^{\infty}\frac{(-1)^k}{k}(-3/4)^k.$$

3. Note that y is the value the series converges to. Therefore, if x doesn't make the series converge (i.e., it's outside the interval of convergence), then there's no y-value that goes with it—in other words, it's outside the domain of the function we're sketching. We see that the graph only has points where x is in the interval of convergence. The interval of convergence for this series is $(-1, 1]$ (just apply the ratio test like we did on the last worksheet). Therefore, our graph only appears on the interval $(-1, 1]$.

   The series in Problem 2 are what you get if you let x equal $-1/3$, 1, -0.1, $-3/4$, and $1/3$, respectively. Based on our answer to Problem 2, it looks like the larger x is, the smaller the series value (and hence the y-value) is. In other words, the function seems to be decreasing. We see that the series in Problem 2, parts (a), (c), and (d) must have positive sums (they're only adding positive terms), so the y-values at $x = -1/3$, $= -0.1$, and $x = -3/4$ must be positive. Looking at the partial sums of the series in Problem 2, parts (b) and (e), we see that when $x = 1/3$ and $x = 1$ the series converges to a negative number, so at these x-values the y-coordinates on our graph must be positive. If we let $x = 0$ in the series, we get $y = \sum(-1)^k 0^k/k = 0$, so the graph must pass through the origin.

   Now what happens to the graph as x approaches the endpoints $x = -1$ and $x = 1$. Well, when $x = 1$, the series converges, so the y-value's some finite value. If x gets close to -1, though, then y "gets close" to $\sum(-1)^k(-1)^k/k = \sum 1/k$, which diverges to $+\infty$—so as x gets close to -1 from the right, the y-value should get arbitrarily large. In other words, we have an asymptote at $x = -1$.

   Putting this all together, we might get a graph like the following:



4. The series equals $\dfrac{1}{1-x}$ when $|x| < 1$.

5. We get

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \qquad \text{and} \qquad \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = -\ln|1-x|.$$

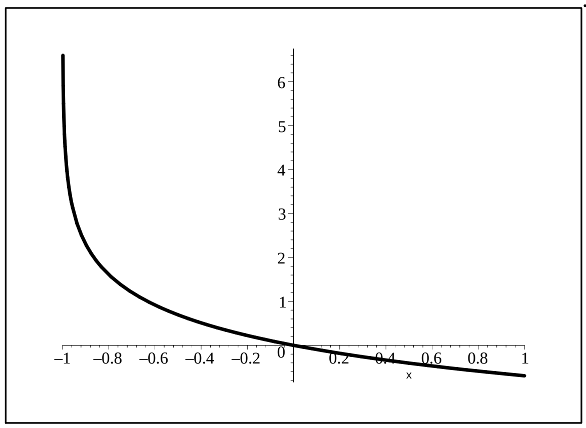
6. See Example 6.6 on page 669 of your text.

7. (a) $1/(1-1/2)^2 = 4$
   (b) $-\ln|1-(-1/3)| = -\ln(4/3)$

8. (a)

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k} \sum_{k=0}^{\infty} \frac{(-x)^{k+1}}{k+1} = -\ln|1-(-x)| = -\ln|1+x|.$$

   (b) The graph of $y = -\ln|1+x|$ is shown below:



The graph you made in problem 3 should be pretty similar to this one—they're the same function.

9. (a) interval of convergence: $(-1, 1)$
   (b) interval of convergence: $(-1, 1)$
   (c) interval of convergence: $[-1, 1)$ (Note that the endpoint is *in* the interval.)

10. (a) interval of convergence: $(-1, 1)$
    (b) interval of convergence: $(-1, 1]$ (One endpoint is in the interval.)
    (c) interval of convergence: $[-1, 1]$ (Both endpoints are in the interval.)

11. The radius of convergence stays unchanged. The interval may change, since whether or not the endpoints are included can change (see the answers to Problems 9 and 10, for example).

13. Hint: in your answer to Problem 6, you came up with a power series which equals $\tan^{-1} x$. Let $x = 1$ in your series–then the series will equal $\tan^{-1}(1)$, which equals $\pi/4$.

Review problems

A. Find $\int \cos x \sin^3 x \, dx$.

B. Does the integral $\int_0^{\infty} \frac{4}{4+x^2} \, dx$ *converge or diverge?*

For Problem A, since the integrand is a product of powers of a sine and a cosine, and since one has an odd exponent (they both do, actually, but one is enough), we use a u-substitution. Either $u = \cos x$ or $u = \sin x$ will do; let's use $u = \sin x$. Then $du = \cos x\, dx$, and

$$\int \cos x \sin^3 x\, dx = \int u^3\, du = \frac{1}{4}u^4 + C = \boxed{\frac{1}{4}\sin^4 x + C}.$$

For Problem B, notice that the integral is improper, since it has an infinite endpoint. We use a comparison test. (Alternatively, we could use a trigonometric substitution on the integral.) Notice that

$$\int_0^\infty \frac{4}{4+x^2}\, dx = 4\int_0^\infty \frac{1}{4+x^2}\, dx, \qquad \text{and} \qquad \frac{1}{4+x^2} < \frac{1}{x^2},$$

and since $\int_1^\infty 1/x^2\, dx$ converges by the p-test (and since $\int_0^1 1/(4+x^2)\, dx$ is finite), we see that

$$\boxed{\int_0^\infty 4/(4+x^2)\, dx \quad \text{converges}}.$$