Selected answers to Merit Worksheet #25

1. $P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3.$

   $R_3(x) = \frac{-(15/16)z^{-7/2}}{4!}(x-4)^4$, where z is between 4 and x.

   $R_3(1/3) = \frac{-(15/16)z^{-7/2}}{4!}(1/3-4)^4$, where z is between 4 and 1/3. Since $z^{-7/2}$ is a decreasing function between 1/3 and 4, the error term is largest if we set $z = 1/3$. Therefore,

$$|R(1/3)| \leq \frac{(15/16)(1/3)^{-7/2}}{4!}(1/3-4)^4.$$

2. (a) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2}x^2 + \frac{x^3}{3!} + \dots$

   (b) $R_n(x) = \frac{e^z}{(n+1)!}x^{n+1}$, where z is between 0 and x.

   (c) As n goes to infinity, both e^z and x are constants. There are a couple of ways to argue it, but the bottom line is that the factorial in the bottom will overwhelm the x^{n+1} in the top, forcing the remainder term to approach zero.

   (d) Since $R_n(x) \to 0$, this means that the error between the Taylor polynomial and the actual function shrinks as you take more terms. In other words, the Taylor *series* for e^x actually does converge to the function e^x

3. (a) The Maclaurin series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots.$$

   When x is very close to 0, we see that x^3, x^5, x^7, etc., are all *very* close to 0, so we can ignore them. That turns the equation above into $\sin x \approx x$.

   (b) Because the second derivative of $\sin x$ is $-\sin x$, which is zero at $x = 0$, the coefficient on x^2 in $P_2(x)$ is 0.

   (c) $R_2(x) = \frac{-\cos z}{3!}x^3$, where z is between 0 and x.

   (d) We see that

$$R_2(\pi/180) = \frac{-\cos z}{3!}(\pi/180)^3,$$

   and since the cosine function is decreasing between 0 and $\pi/180$, with its maximum value equal to 1 (which happens at $z = 0$), we see that

$$|R_2(\pi/180)| \leq \frac{1}{3!}(\pi/180)^3.$$

4. Look these up in your book.

5. (a)

$$\frac{1}{1+3x} = \sum_{k=0}^{\infty}(-3x)^k = \sum_{k=0}^{\infty}(-3)^k x^k.$$

   Interval of convergence: $(-1/3, 1/3)$

(b)

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k}.$$

Interval of convergence: $(-\infty, \infty)$

(c)

$$\frac{\sin x}{x} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k}.$$

Interval of convergence: $(-\infty, \infty)$

(d)

$$\frac{1-\cos x}{x} = \frac{1-(1-x^2/2!+x^4/4!-x^6/6!+\dots)}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!} x^{2k-1}.$$

Interval of convergence: $(-\infty, \infty)$

(e) We haven't learned any shortcuts yet that will help us on this one. We will, later on, when we learn the binomial series. For now, though, you'll just want to find the Taylor series for this function as you would any other function you don't know a shortcut for–build the series $\sum f^{(k)}(c)/k!x^k$ from scratch. The answer you get starts off

$$\sqrt{1-x^2} = 1 - 1/2\,x^2 - 1/8\,x^4 - 1/16\,x^6 - \frac{5}{128}x^8 - \frac{7}{256}x^{10} - \frac{21}{1024}x^{12} - \frac{33}{2048}x^{14} - \frac{429}{32768}x^{16} - \dots.$$

As you might guess, looking at the series, and as we'll see later, the interval of convergence is $(-1, 1)$.

6. (a)

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1, \qquad \text{and} \qquad \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0.$$

(b) The series for $(\sin x)/x$ has a constant term equal to 1, so plugging 0 into that series gives you the same value as the limit. The series for $(1-\cos x)/x$ is 0, so plugging 0 into the series gives you the same value as the limit.

7.

$$\frac{1}{(1+3x)^2} = \sum_{k=1}^{\infty} (-3)^{k-1} k x^{k-1}.$$

The series will have the same radius of convergence as the series for $1/(1+3x)$ did, namely, $1/3$.

8.

$$f(x) = \int_0^x e^{-t^2}\,dt = \int_0^x \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{2k} \right) dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} x^{2k+1} dx.$$

The interval of convergence is $(-\infty, \infty)$.

9. The actual value is 0.7468241330...

Review problems

Fill in the blanks with "converges," "diverges," or "may or may not converge—we can't tell yet."

A. If $\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1/2$, then $\sum_{k=1}^{\infty} a_k$ **CONVERGES**, by the root test.

B. If $\lim_{k\to\infty} k^2 a_k = 2$, then $\sum_{k=1}^{\infty} a_k$ **CONVERGES**, by the limit comparison test with the series $\sum 1/k^2$.