Selected answers to Merit Worksheet #26

1. $-1/6$
2. (a) 1

   (b) The 8th degree polynomial is $P_8(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$. The integral of this polynomial from 0 to 1 equals

   $$\left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \frac{x^9}{9\cdot 9!}\right]_0^1 = 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} + \frac{1}{9\cdot 9!}.$$

   (c) Using a calculator, we find that the expression above equals 0.9460830726. Notice that this agrees with the exact value up to 8 decimal places.
3. (a) $-1/2$

   (b) $1/6$
4. The Taylor series for $\ln x$ about $x = 1$ is

   $$\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k,$$

   valid for x in $(0, 2]$. Letting $x = 2$, we get

   $$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k},$$

   so the alternating harmonic series converges to $\ln 2$.
5. The Taylor polynomial we'll use is for the function e^x:

   $$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, \qquad \text{so} \qquad e^{\sqrt{x}} \approx 1 + \sqrt{x} + \frac{x}{2} + \frac{x^{3/2}}{6}.$$

   Therefore,

   $$\int_0^1 e^{\sqrt{x}}\,dx \approx \int_0^1 \left(1 + \sqrt{x} + \frac{x}{2} + \frac{x^{3/2}}{6}\right) dx = 119/60 = 1.98333...$$

   (Actually, the integral is doable, and its value is 2.)
6. The interval of convergence is $(0, 2]$. $\ln 3$ can be approximated in many ways. We can't just set $x = 3$ in the answer to Problem 4, because 3 is not in the interval of convergence, but we can do the following:

   $$\ln 3 = \ln(1/3)^{-1} = -\ln(1/3);$$

   Since 1/3 IS in the interval of convergence, we know that

   $$\ln(1/3) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(1/3-1)^k.$$

   Therefore, $\ln 3 = -\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(1/3-1)^k$. To approximate $\ln 3$, we may just take the first couple of terms. The first three give

   $$\ln 3 \approx 2/3 + \frac{2}{9} + \frac{8}{81} + \frac{4}{81} = \frac{80}{81} = 0.9876543210$$

   (Actually, this isn't a very good approximation, but it's better than nothing.)

7. (a) We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{k=0}^{\infty} x^k.$$

whenever $|x| < 1$. Therefore,

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

whenever $|x| < 1$. Integrating now, we find

$$\arctan x = \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots \right) + C = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + C.$$

Now to find out what the constant of integration C is, we let $x = 0$ and find that $C = 0$. Therefore,

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

whenever $|x| < 1$. Finally, we find that

$$\int_0^1 \arctan x \, dx = \left[\frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{30}x^6 - \frac{1}{56}x^8 + \frac{1}{90}x^{10} - \dots \right]_0^1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+2)} x^{2k+2}.$$

Since we only need an approximation of $\int_0^1 \arctan x$, let's just sum the first four terms:

$$\int_0^1 \arctan x \, dx \approx \frac{1}{2} - \frac{1}{12} + \frac{1}{30} - \frac{1}{56} = 0.432...$$

(b) We can apply an integration by parts with $u = \arctan x$ and $dv = dx$:

$$\int_0^1 \arctan x \, dx = x \arctan x|_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx = \arctan 1 - \left[\frac{1}{2} \ln(1+x^2) \right]_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

Review problems

A. *Determine whether or not the series converges:* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. The series converges by the integral test.

B. *Find the integral* $\displaystyle\int_1^e x \ln x \, dx$. Use integration by parts, with $u = \ln x$ and $dv = x\,dx$. The answer is $\dfrac{1}{4}e^2 + \dfrac{1}{4}$.