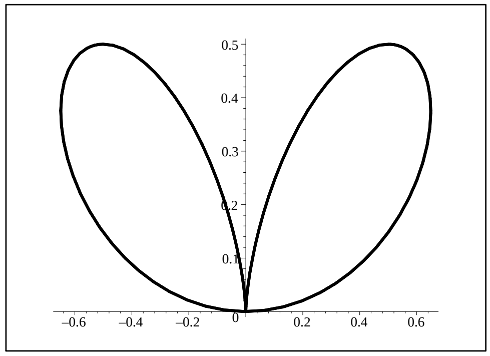
Merit Worksheet #35, 4/25/08

A bit more of calculus and polar curves

1. Shown below is the graph of $r = \sin 2\theta \cos\theta$.



   (a) If θ starts at 0, where should we *stop* θ so that the curve is traced out exactly once between the starting and stopping points?

   (b) Label the graph with arrows at a few points to make it clear in which direction the graph is traced out.

   (c) What is the slope of the tangent line to the graph when $\theta = \pi/2$? When $\theta = 2\pi/3$?

   (d) Which point or points on the curve are the farthest away from the origin? (For each point, give both polar coordinates, or both cartesian coordinates.)

   (e) How much area is enclosed by one loop of the graph?

2. Sketch the graph of $r = 1 + \cos\theta$. This graph is called a *cardioid*.

   (a) On your graph, indicate in which direction the graph is traced out.

   (b) Find all points where the tangent line to the curve is horizontal or vertical. (For each point, give both polar coordinates, or both cartesian coordinates.)

   (c) Find the area enclosed by this graph.

3. Find the area inside the graph of $r = 2\sin\theta$ but outside the graph of $r = 1$.

Conic sections

4. What exactly is a conic section? There are three (or four or five, depending on how you classify things) basic conic sections. What are they?

5. (a) Give a geometric definition of a parabola. (See page 764 in your text.)

   (b) Using only this geometric definition and the distance formula, find the equation of the parabola with directrix $y = -1$ and focus $(1, 1)$.

   (c) Find a polar equation and bounds on θ that trace out the parabola $y = x^2$.

6. (a) Give a geometric definition of an ellipse. (See page 767 of your text.)

   (b) Using a piece of cardboard, two thumbtacks, a piece of string, and a pencil, how can you construct an ellipse?

(c) Why is a circle considered an ellipse? In other words, a circle is an ellipse with *what* special property?

7. (a) Give a geometric definition of a hyperbola. (See page 770 in your text.)

   (b) Suppose the energy from an earthquake reaches one seismic observation station 20 seconds sooner than it reaches another (and we know how fast earthquake energy travels, which is roughly constant). How can we use this information to determine where the earthquake's epicenter is located? Why is this information *not* enough to determine the exact location of the epicenter? Would knowing how long it took the earthquake to reach a third station help?

8. (a) Find an equation of the form $r = f(\theta)$ that traces out the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

   (b) Show that the parametric equations $x = A\cos t$, $y = B\sin t$, $0 \le t \le 2\pi$ trace out an ellipse. (Hint: show that the curve satisfies the equation in part (a) if you choose a and b carefully.)

   (c) The parametric equations $x = A\cosh t$, $y = B\sinh t$, $-\infty < t < \infty$ trace out a portion of *which* conic section?[1]

9. Summarize briefly the light reflection properties of ellipses, parabolas, and hyperbolas, as described on pages 766–767 (parabolas), 769–770 (ellipses), and 772 (hyperbolas). Then visit the ellipse on the south side of the quad, in front of Foellinger Auditorium, and be amazed (or at least read about it in Wikipedia's article on "ellipse").

Preparation for next time

With this worksheet, we've covered the last of the material we'll cover during the semester. On Monday and Wednesday of next week, we'll review for the final. In preparation for Monday's class, please look over the final review problems that have been attached to the past several worksheets. Come to class prepared to ask questions on problems that you found challenging, or on topics that you'd like to see reviewed. There will be nothing to turn in.

Mathematical sidenote of the day

You've noticed by now that I've been switching the groups every chance I can. (There's a pedagogical reason for this; I'm not just doing it on a whim.) As I've mentioned, I use a computer to assign the groups in a somewhat random manner. How random? *Truly* random.

You might ask how it's possible to get truly random numbers with a computer. After all, someone had to program the computer to generate the numbers, so how can they be really random? How can you design a purely random outcome? Even flipping a coin doesn't work perfectly—it's been shown recently that when you flip a coin, the coin is slightly more likely to land oriented the way it was before you flipped it. So what's a randomness-hungry person to do?

Enter `www.random.org`, an online random number generating service. The way `random.org` generates its random numbers is to let nature do it; according to the website,

"RANDOM.ORG offers true random numbers to anyone on the Internet. The randomness comes from atmospheric noise, which for many purposes is better than the pseudo-random number algorithms typically used in computer programs ... [True random number generators] extract randomness from physical phenomena and introduce it into a computer. You can imagine this as a die connected to a computer, but

[1] The answer to part (c), along with its similarities to part (b), explain in part why we call these the *hyperbolic trigonometric functions*.

typically people use a physical phenomenon that is easier to connect to a computer than a die is... [A] suitable physical phenomenon is atmospheric noise, which is quite easy to pick up with a normal radio. This is the approach used by RANDOM.ORG... Regardless of which physical phenomenon is used, the process of generating true random numbers involves identifying little, unpredictable changes in the data. For example, ... RANDOM.ORG uses little variations in the amplitude of atmospheric noise."

An article in *Science News* says the following:

"Random.org uses a radio to pull random numbers out of the atmospheric noise generated by weather systems.

" 'When we built this in 1997, we bought the cheapest radio we could find,' says Mads Haahr, of Trinity College, Dublin, who runs Random.org. 'The guy in the shop thought we were crazy because we made him take it out of the box and put in batteries so we could listen to the noise between stations.'

"Haahr's Web site (http://www.random.org/) can generate up to 3,000 random numbers per second. Over the last 6 years, it has dished out more than 61 billion random numbers for free to an eclectic array of users. These include archaeologists choosing which quadrants of a large area to survey; a choreographer selecting the order, timing, and placement of dance steps; online card-playing sites shuffling their virtual decks; the U.S. Environmental Protection Agency determining which companies to include in a random audit of hazardous-material use; and a locksmith deciding how deeply to cut the notches on keys."

And, of course, your Math 231 instructor, in determining how to arrange you into groups.

If you'd like to read more about random.org or other random number generators, please visit one of the many articles linked to at `http://www.random.org/media/`.

Review problems for the final — Sections 8.8 and 9.1

A. Use at least three terms of an appropriate Taylor series to approximate $\sqrt{65}$.

B. Without using l'Hospital's rule, conjecture the value of the following limit:

$$\lim_{x \to 0} \frac{x - x^3/6 - \sin x}{2x^5}.$$

C. Use a Taylor polynomial with at least five terms to approximate $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$.

D. Find the first five terms of the Maclaurin series for the following functions:

$$\text{(a)}\quad \frac{1}{1+x^2} \qquad \text{(b)}\quad (1-x)^{1/3} \qquad \text{(c)}\quad \frac{1}{\sqrt{1-x^2}}$$

E. Using some of your answers to Question D above, find the first five terms of the Maclaurin series for $\tan^{-1} x$ and $\sin^{-1} x$.

F. (a) Suppose a curve is given parametrically by $x(t) = 2t - 1$, $y(t) = 4t^2 - 5$. Eliminate the parameter to find an equation for the curve only in terms of x and y.

(b) Do the same for the curve $x = 3\cos 2t$, $y = 2 + \sin 2t$.

G. Two objects have their positions given by

$$\begin{cases} x = 4 - t, \\ y = (t-1)^2, \end{cases} \qquad \text{and} \qquad \begin{cases} x = s, \\ y = s - 1. \end{cases}$$

Find all points of intersection of the two paths. Also, assuming that s and t are measured from the same time and on the same scale, determine if the objects ever collide, and, if so, when and where.