Name: SOLUTIONS

Math 231 W3, Spring Term 2009
Mock Exam #2
March 12, 2009

No books, notes, calculators, or other aids may be used. For full credit you must show all your work on each problem.

Problem	Score	Points Possible
1		20
2		15
3		20
4		25
5		20
TOTAL		100

Problem 1: (20 points) (a) Is either of $y = t^2 + 6t + 9$ or $y = te^{-3t}$ a solution to the differential equation $y'' + 6y' + 9y = 0$? Justify your answer, showing all your work.

We'll check each possibility:

If $y = t^2 + 6t + 9$

then $y' = 2t + 6$

$y'' = 2$

So $y'' + 6y' + 9y = 2 + 6(2t+6) + 9(t^2+6t+9)$

$= 9t^2 + 66t + 119$

$\neq 0$

so this y is NOT a solution to the differential equation.

If $y = te^{-3t}$, then

$y' = e^{-3t} - 3te^{-3t}$

$y'' = -6e^{-3t} + 9te^{-3t}$

So $y'' + 6y' + 9y = -6e^{-3t} + 9te^{-3t} + 6e^{-3t} - 18te^{-3t} + 9te^{-3t}$

$= 0$

so $y = te^{-3t}$ IS a solution to the differential equation.

(b) A bacteria culture is known to grow so that its population $y(t)$ satisfies the differential equation

$$y'(t) = 0.23y(t).$$

Find the solution to this equation, given that there were 100 bacteria at time $t = 0$.

Since the equation has the form $y' = ky$, we know the solution has the form $y(t) = Ae^{kt}$.

Here, $y(t) = Ae^{0.23t}$.

We also know $y(0) = 100$, so

$100 = Ae^{0.23 \cdot 0}$

$= A$

So the solution is $\boxed{y(t) = 100e^{0.23t}}$.

Problem 2: (15 points) (a) For each of the following sequences $\{a_k\}_{k=1}^{\infty}$, state whether the sequence converges or diverges, and give the limit of any that converge.

(i) $a_k = \dfrac{\sqrt{k}}{k^3+1}$

l'Hôpital's rule　　　algebra

$$\lim_{k\to\infty} \frac{\sqrt{k}}{k^3+1} = \lim_{k\to\infty} \frac{k^{1/2}}{k^3+1} = \lim_{k\to\infty} \frac{\frac{1}{2}k^{-1/2}}{3k^2} = \lim_{k\to\infty} \frac{1/2}{3k^{5/2}}$$

$$= 0.$$

So the sequence $\boxed{\text{converges to } 0.}$

(ii) $a_k = (-1)^k \dfrac{k^2-k}{k^2+1}$

Since $\lim_{k\to\infty} \frac{k^2-k}{k^2+1} = 1$, but the $(-1)^k$ keeps changing the terms from positive to negative and vice versa, the sequence $\boxed{\text{diverges.}}$

(b) What does it mean for a sequence to be monotonic?

A sequence is monotonic if it is either always increasing or always decreasing.

Problem 3: (20 points) Determine whether the series below converge or diverge. Clearly state any convergence/divergence tests you use. For those series that converge, find their limit.

(a) $\sum_{k=1}^{\infty}(-1)^k \frac{5}{2^{k-2}}$ This is a geometric series.

$$= \sum_{k=1}^{\infty} \frac{5(-1)^k}{2^{-2} \cdot 2^k} = \sum_{k=1}^{\infty} 5 \cdot 2^2 \left(\frac{-1}{2}\right)^k$$

since $r = -\frac{1}{2}$ and $|r| < 1$, the series $\boxed{\text{Converges}}$ to

$$\frac{\text{first term}}{1-r} = \boxed{\frac{-10}{1-(-\frac{1}{2})} = \frac{-20}{3}}$$

(b) $\sum_{k=2}^{\infty} \frac{1}{k-1}$

Since $\frac{1}{k-1} > \frac{1}{k}$, $\sum_{k=2}^{\infty} \frac{1}{k-1}$ $\boxed{\text{diverges}}$ by the $\boxed{\text{Comparison Test (with the harmonic series)}}$

ALSO POSSIBLE: • L.C.T. with the harmonic series
• Integral Test

(c) $\sum_{k=1}^{\infty}(\ln(k+1) - \ln k)$ (Hint: this series is a telescoping series.)

The n^{th} partial sum is

$$S_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \dots + (\ln n - \ln(n-1)) + (\ln(n+1) - \ln n)$$

$$= -\ln 1 + \ln(n+1)$$

As $n \to \infty$, $S_n \to \infty$ since $\lim_{n\to\infty} [-\ln 1 + \ln(n+1)] = \infty$.

Therefore, the series $\boxed{\text{diverges.}}$

Problem 4: (25 points) Determine whether each of the series below converges absolutely, converges conditionally, or diverges. Clearly state any convergence/divergence tests you use, and show all your work.

(a) $\sum_{k=0}^{\infty} \frac{10k^2}{k^3+1}$ By the Limit Comparison Test (with $\sum \frac{1}{k}$),

since $\lim_{k\to\infty} \frac{10k^2/(k^3+1)}{1/k} = \lim_{k\to\infty} \frac{10k^2}{k^3+1}\cdot\frac{k}{1} = \lim_{k\to\infty} \frac{10k^3}{k^3+1} = 10$,

and since $\sum \frac{1}{k}$ diverges, the series $\boxed{\text{DIVERGES.}}$

(b) $\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$ ← It's an alternating series. Since

$\lim_{k\to\infty} \frac{1}{k\ln k} = 0$ and $\frac{1}{k\ln k}$ decreases as $k\to\infty$,

the Alternating Series Test says this series $\boxed{\text{CONVERGES.}}$ However, $\sum \left|\frac{(-1)^k}{k\ln k}\right| = \sum \frac{1}{k\ln k}$, which diverges by the Integral Test, so $\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$ converges $\boxed{\text{CONDITIONALLY.}}$

(c) $\sum_{k=0}^{\infty} \frac{(-10)^k(k+3)}{k!}$ We use the Ratio Test.

$$\lim_{k\to\infty} \left| \frac{(-10)^{k+1}[(k+1)+3]}{(k+1)!} \cdot \frac{k!}{(-10)^k(k+3)} \right| = \lim_{k\to\infty} \frac{10^{k+1}}{10^k}\cdot\frac{k+4}{k+3}\cdot\frac{\cancel{k(k-1)\cdots 3\cdot 2\cdot 1}}{(k+1)\cancel{k(k-1)\cdots 3\cdot 2\cdot 1}}$$

$$= \lim_{k\to\infty} 10\cdot\frac{k+4}{k+3}\cdot\frac{1}{k+1}$$

$$= 0$$

So the series $\boxed{\text{CONVERGES ABSOLUTELY.}}$

Problem 5: (20 points) (a) Mark the following statements as true or false. If a statement is false, write a corrected statement underneath it.

(i) If $\lim_{k \to \infty} a_k = 0$, then $\sum_{k=0}^{\infty} a_k$ converges.

FALSE. If $\lim_{k \to \infty} a_k \neq 0$, then $\sum_{k=0}^{\infty} a_k$ diverges.

(OR: If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k \to \infty} a_k = 0$.)

(ii) If $\lim_{k \to \infty} \sqrt[k]{|a_k|} = 1$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

FALSE. If $\lim_{k \to \infty} \sqrt[k]{|a_k|} = 1$, the Root Test tells us nothing about whether $\sum_{k=1}^{\infty} a_k$ converges.

(iii) The integral test and comparison tests can only be applied to series having terms which are not negative.

TRUE.

(b) (i) Mark on the number line below the approximate locations of the first 6 partial sums of the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$.

(ii) How far away could S_6 (the 6th partial sum) possibly be from the total sum of the series? (There should not be any variables in your final answer, but you do not need to simplify beyond that point.)

S_6 differs from the sum $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ by no more than a_7, which is $\boxed{\frac{1}{7^2} = \frac{1}{49}}$.