Solutions to Math 231 W3 Written Assignment #1

1. **Define a differential equation, and the solution to a differential equation.**

A differential equation is any equation involving any derivative of an unknown function. A solution to a differential equation is a function that makes the differential equation true when you plug it in.

2. **Which of the following are solutions to the differential equation**

$$y''(x) - 4y'(x) + 4y(x) = 0?$$

(There may be more than one solution shown.)

(a)	$y(x) = e^{2x}$	(c)	$y(x) = 0$	(e)	$y(x) = x^2 - 4x + 4$
(b)	$y(x) = \sin 2x + \cos 2x$	(d)	$y(x) = xe^{2x}$	(f)	$y(x) = 3e^{2x} - 4xe^{2x}$

If $y(x) = e^{2x}$, then

$$y'' - 4y' + 4y = 4e^{2x} - 8e^{2x} + 4e^{2x} = 0.$$

If $y(x) = \sin 2x + \cos 2x$, then

$$y'' - 4y' + 4y = (-4\sin 2x - 4\cos 2x) - 4(2\cos 2x - 2\sin 2x) + 4(\sin 2x + \cos 2x) = 8\sin 2x - 8\cos 2x.$$

If $y(x) = 0$, then

$$y'' - 4y' + 4y = 0 - 0 + 0 = 0.$$

If $y(x) = xe^{2x}$, then

$$y'' - 4y' + 4y = (4e^{2x} + 4xe^{2x}) - 4(e^{2x} + 2xe^{2x}) + 4xe^{2x} = 0.$$

If $y(x) = x^2 - 4x + 4$, then

$$y'' - 4y' + 4y = (2) - 4(2x - 4) + 4(x^2 - 4x + 4) = 4x^2 - 24x + 34.$$

If $y(x) = 3e^{2x} - 4xe^{2x}$, then

$$y'' - 4y' + 4y = (-4e^{2x} - 16xe^{2x}) - 4(2e^{2x} - 8xe^{2x}) + 4(3e^{2x} - 4xe^{2x}) = 0.$$

Thus, the solutions are (a), (c), (d), and (f).

3. **Suppose we're given the differential equation**

$$y'' + y' + 2y = 4t^3 - 4t^2 - 3,$$

and we're told that one of the solutions has the form $y = At^3 + Bt^2 + Ct + D$, where A, B, C, and D are unknown constants. What should A, B, C, and D be in order for y to be a solution to the differential equation?

Since

$$\begin{array}{rcccccccc} y &=& At^3 &+& Bt^2 &+& Ct &+& D, \\ y' &=& & & 3At^2 &+& 2Bt &+& C, \\ y'' &=& & & & & 6At &+& 2B, \end{array}$$

we find that

$$y'' + y' + 2y = 2At^3 + (3A + 2B)t^2 + (6A + 2B + 2C)6 + (2B + C + 2D).$$

However, we also know that $y'' + y' + 2y = 4t^3 - 4t^2 - 3$, so we can set the coefficients in front of the different powers of t equal to each other. Setting the coefficients on the t^3's equal, we get $A = 2$. The coefficients on the t^2's tell us that $3A + 2B = -4$, which tells us that $B = -5$. The coefficients on t tell us that $6A + 2B + 2C = 0$, which, together with what we know about A and B, tells us that $C = -1$. Finally, $2B + C + 2D = -3$, so $D = 4$. We've therefore found that

$$y(t) = 2t^3 - 5t^2 - t + 4$$

is a solution to the differential equation.

4. **What is the solution to the differential equation $y'(t) = ky(t)$?**

This particular differential equation has the solution

$$y(t) = Ae^{kt}$$

(which we've memorized), where the k in the exponent is the same as the k in the equation, and A is just a constant.

5. **Find the solution to the following differential equations satisfying the given initial conditions.**

   (a) $y' = -2$, $y(0) = -8$

   (b) $y' = 4y$, $y(0) = 2$

   (c) $y' = 2y$, $y(1) = 2$

For (a), we can just integrate. If $y'(t) = -2$, then integrating tells us that $y(t) = -2t + C$, where C is an unknown constant. Since $y(0) = 8$, we get

$$8 = -2 \cdot 0 + C,$$

so $C = 8$. That means that

$$\boxed{y(t) = -2t + 8.}$$

Now for (b) and (c). Whenever we see the equation $y' = ky$, we know that the solution has the form $y(t) = Ae^{kt}$, where the k in the exponent is the same as the k in the equation. So in part (b), we start out knowing that the solution looks like $y(t) = Ae^{4t}$. We don't know what A is yet, though, so we use the second equation. When $t = 0$, $y = 2$, so we plug that in and get

$$2 = Ae^{4 \cdot 0},$$

which, when we simplify it, tells us that $A = 2$. That means that

$$\boxed{y(t) = 2e^{4t},}$$

which is our solution.

Along those same lines, for (c) we find out that $y = Ae^{2t}$, but when we plug in $y(1) = 2$, we get

$$2 = Ae^{2 \cdot 1},$$

so $A = 2/e^2$. The solution is therefore

$$\boxed{y(t) = \frac{2}{e^2}e^{2t}.}$$

6. **Radioactive decay satisfies the differential equation**

$$y'(t) = -ky(t).$$

Suppose a radioactive substance has a half-life of 5 days (i.e., it takes 5 days for half of the substance to decay). If 1000 **grams of the material are placed under observation, how much will remain after 28 days?**

We know that the solution to the differential equation must have the form $y(t) = Ae^{-kt}$. We need to know what A and k are. We know that $y = 1000$ when $t = 0$, so

$$1000 = Ae^{-k\cdot 0} = A.$$

We also know that after 5 days half that amount will be left, so

$$500 = 1000e^{-k\cdot 5}.$$

Solving for k, we find

$$k = -\frac{\ln(1/2)}{5},$$

so

$$y = 1000e^{[\ln(1/2)]/5\cdot t}.$$

To see how much remains after 28 days, we plug in $t = 28$ to find out $y = 1000e^{[\ln(1/2)]/5\cdot 28} \approx 20.6$, i.e., about 20.6 grams.

7. **Bacterial growth can be modeled by the differential equation**

$$y'(t) = ky(t).$$

Say a bacterial culture triples in population every 2 hours. If we're told that the bacterial culture started out from a single cell and now has 400 cells, how long has the culture been growing?

Once again, we know that the solution to the differential equation looks like $y(t) = Ae^{kt}$. Since the initial population was 1 cell, we know that $A = 1$. We also know that after 2 hours, the population has tripled, so

$$3 = 1e^{k\cdot 2}.$$

Solving for k, this gives us $k = (\ln 3)/2$, so $y = 1e^{[(\ln 3)/2]t}$. We're told that the population now is 400 cells, and we want to know what t is. Plugging this in, we've got to solve

$$400 = 1e^{[(\ln 3)/2]t}.$$

We find that $t = 2(\ln 400)/(\ln 3) \approx 10.9$ hours.