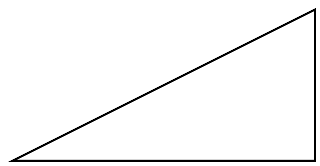
Merit Worksheet #4, 1/28/09

Trigonometric substitutions

1. Suppose you're told that $x = 3\tan\theta$. Label the triangle below to illustrate that equation, and use the triangle to find an expression for $\cos\theta$.



2. Consider the integral

$$\int \frac{1}{\sqrt{1-x^2}}\,dx,$$

and suppose you've forgotten what the answer to this one is; we're going to use a trigonometric substitution to find it.

   (a) Your book suggests that since the integral involves a quantity of the form $a^2 - x^2$, you should make the substitution $x = a\sin\theta$. In this problem, what would x and dx equal?

   (b) Substitute x and dx into the integral, and find the antiderivative.

   (c) ***Why*** was it a good idea to make the substitution suggested?

3. Use trigonometric substitutions to find the following integrals:

   (a) $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$ (b) $\displaystyle\int \frac{\sqrt{x^2-4}}{x^2}\,dx$ (c) $\displaystyle\int \frac{dx}{x^2\sqrt{1-4x^2}}$

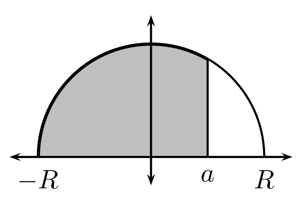
4. Consider the integral

$$\int_0^1 x^3\sqrt{1+x^2}\,dx.$$

   (a) Find the integral through a u-substitution.

   (b) Find the integral through a trigonometric substitution.

5. Find the area enclosed by the x-axis, the upper half of the circle of radius R centered at the origin, and the line $x = a$, where $-R < a < R$.



6. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

What happens if both a and b equal R?

7. Find the following integrals

(a) $\displaystyle\int \frac{dx}{(1-x^2)^{3/2}}$ (b) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$ (c) $\displaystyle\int \frac{dx}{x^2+4x+13}$

8. Hopefully you've seen how certain trigonometric identities can be used to make substitutions that really simplify an integral. Now let's look at another technique that's often used:

   In case you've never met them before, the *hyperbolic sine* and *hyperbolic cosine* functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

   These functions show up a lot in engineering applications. One reason for their names is that they satisfy the following trig-like identity:

$$\cosh^2 t - \sinh^2 t = 1.$$

   (a) If we were to take advantage of the identity above, what substitution would we make if our integral had an $x^2 + a^2$ in it? What about $x^2 - a^2$?

   (b) Use hyperbolic substitutions to find the following integrals:

   (a) $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$ (b) $\displaystyle\int \frac{\sqrt{x^2-4}}{x^2}\,dx$

Reading assignment for Friday, 1/30: Next time we'll have a quiz on Sections 6.2 and 6.3; come prepared for that. After that we will go over how to integrate integrals with quadratic denominators; you actually already know everything you need to solve these problems, but it will be important to review it before moving on to Section 6.4. So the reading assignment comes from **back in Section 6.1.** I'd like you to read through Examples 1.4 and 1.5 on pages 511-512, paying attention to the details of the problems (ugly as they may seem at first). If you need a review on completing the square, I'd recommend the following two example videos online:

`http://www.youtube.com/watch?v=A9s1RtFls64`
`http://www.youtube.com/watch?v=GV2OVD7bKKY`

Then work exercise 23 in Section 6.1, and, as always, write a question about what you've read (or watched).

Quotes of the day:
"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is." —John von Neumann

"Mathematics are well and good but nature keeps dragging us around by the nose." —Albert Einstein

"Nature laughs at the difficulties of integration." —Pierre-Simon de Laplace