

Merit Worksheet #6, 2/04/09

Partial fractions

1. Find the integrals (a) $\int \frac{2x+2}{x^2-2x} dx$ and (b) $\int \frac{x}{(x-1)^2} dx$.
2. The process of breaking a rational function into its smallest “pieces” is called finding the *partial fraction decomposition*. From what you’ve seen so far, write down on the board what you’d you say are the steps of finding a partial fraction decomposition.
3. Set up the *form* of the partial fractions for each rational function (but do not finish finding the actual partial fraction decomposition).

Example:

$$\frac{x^2}{(x-1)^2(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$(a) \frac{2x+2}{x^2-2x} \quad (b) \frac{x^2-4x+1}{(x+1)(x-2)^2} \quad (c) \frac{5x^3-5x^2+x-2}{x^2(x^2+1)} \quad (d) \frac{1}{(x^4-1)^3}$$

4. Suppose you’re finding a partial fraction decomposition, and you’ve written

$$\frac{7x-6}{(x+1)(2-3x)} = \frac{A}{x+1} + \frac{B}{2-3x}.$$

You multiply both sides of the equation by $(x+1)(2-3x)$ (the denominator of the left-hand side) to get

$$7x-6 = A(2-3x) + B(x+1).$$

You wish to substitute into this equation values of x that will make the resulting equation very simple. What values of x would do that? What are the equations (with A and B) you would get?

Example: Plugging in $x = 5$, you get

$$29 = -13A + 6B.$$

You can get simpler equations if you choose just the right x ’s. Which x ’s are they?

5. Find the following integrals:

$$(a) \int \frac{5x-2}{x^2-4} dx \quad (b) \int \frac{x^2-4x+1}{(x+1)(x-2)^2} dx \quad (c) \int \frac{x^5+6x^3-4x^2+x-2}{x^2(x^2+1)} dx$$

6. Consider the integral

$$\int \frac{4x}{x^4+4} dx.$$

- (a) Use partial fractions to solve the integral. Hint: It may help to write

$$x^4+4 = (x^4+4x^2+4) - 4x^2.$$

- (b) Wondering if there's an easier way to find this integral than the way you did it in part (a), you look to see if there's a u -substitution that would make the integral simpler. Can you find one? If so, do you get the same answer in this way that you did in part (a)?

7. Find the integral $\int \frac{1}{x^3 - 1} dx$. (Hint: the denominator can be factored.)

8. Glancing in the back of some textbook, you find the following formula:

$$\int \frac{u^2}{a^2 - u^2} du = -u + \frac{a}{2} \ln \left| \frac{a+u}{a-u} \right| + C.$$

Derive this formula.

9. Suppose you needed to find a whole bunch of derivatives of the function

$$y = \frac{1}{x^2 - 1}$$

at the point $x = a$.*

- Using the chain and product rules as necessary, find the first three derivatives of $y = (x^2 - 1)^{-1}$.
 - Find the partial fraction decomposition of $\frac{1}{x^2 - 1}$.
 - Now using your answer to part (b), can you find a formula for the n th derivative of y , no matter what n is?
10. And now, finally, we see another way to find $\int \sec x dx$! Starting from the steps

$$\int \sec x dx = \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \frac{1}{1 - \sin^2 x} \cos x dx,$$

use a u -substitution and partial fractions to arrive at an antiderivative. Neat, huh?

Reading assignment for Friday, 2/6: Sections 6.1 through 6.4 contain all the integration techniques we'll learn in this course. The thing to do now is to practice them and learn when to use each. For your preparation assignment for next time, there will be no extra reading. Instead, look at problems 1-44 of the Review Exercises beginning on page 561. From this list, write out the solution for (1) a problem using u -substitution or integration by parts, (2) an integral involving powers of trig functions or a trigonometric substitution, and (3) an integral whose solution involves partial fractions. Also, as usual, write down a question—this time it can be about anything we've covered so far.

(I would **STRONGLY** encourage you to try to work all of problems 1-50 in preparation for the exam on February 16. The exam will contain a bunch of integrals very similar to these.)

Quote of the day:

“A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.”

—Count Lev Nikolgeevich Tolstoy (1828-1920)

*When we learn about Taylor series in Chapter 8, you'll see why we might need to do this.