Merit Worksheet #8, 2/9/09

Improper integrals

1. Which of the following integrals are *improper?* What makes each one improper? (No cheesy puns here, please.)

   (a) $\displaystyle\int_1^e \ln x\, dx$ (b) $\displaystyle\int_2^\infty \frac{dx}{x\ln x}$ (c) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2}$

   (d) $\displaystyle\int_{-1}^1 \frac{dx}{x^2}$ (e) $\displaystyle\int \sin\left(\frac{1}{x}\right) dx$ (f) $\displaystyle\int_a^b \tan x\, dx$

2. Determine whether or not the improper integrals converge. If they converge, find their value:

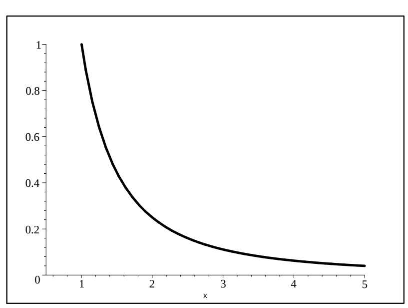
   (a) $\displaystyle\int_0^\infty e^{-x}\, dx$ (b) $\displaystyle\int_0^1 \frac{1}{x^{3/2}}\, dx$ (c) $\displaystyle\int_0^2 \frac{x}{x^2-1}\, dx$

3. The integral
$$\int_0^\infty \frac{1}{x^{1/2}+x^{3/2}}\, dx$$
is improper because of *both* its endpoints. Write this integral as the sum of two integrals—one from 0 to 1, the other from 1 to ∞, and determine whether or not it converges. If it converges, find its value. (Hint: In finding the antiderivative, make the substitution $u = \sqrt{x}$.)

4. The integral
$$\int_{-\infty}^\infty \frac{x}{(x^2+4)^{3/2}}\, dx$$
is also improper because of both its endpoints. Does it converge? If so, to what? (You may want to look at page 553 and the warning on page 554 of your text.)

5. Find all real number values of k for which
$$\int_1^\infty \frac{1}{x^k}\, dx$$
converges. Evaluate the integral for those values of k.

6. Find
$$\int_1^\infty \frac{dx}{x^2}.$$
Does this say that the area under $y = 1/x^2$, $x \geq 1$, is finite? How can that be, when the region described is infinitely long?



7. If you go on to take a differential equations course (and I suspect some of you will), you'll learn about the *Laplace transform.* The Laplace transform of a function $f(x)$ is defined as the new function $F(s)$ given* by

$$F(s) = \int_0^\infty e^{-sx} f(x)\, dx.$$

Find the Laplace transform of the following functions, and give the domain of F (i.e., for what s does the integral converge?).

(a) $f(x) = 1$

(b) $f(x) = \cos 2x$

(c) $f(x) = x$

(d) $f(x) = e^{ax}$

8. **Escape velocity.** In Jules Verne's 1865 classic *From the Earth to the Moon*, a manned space ship is fired into space from a cannon. Under such conditions, where the space ship itself has no means of propulsion, the space ship has to be fired at a high enough speed so that it escapes the earth's gravity—the problem, though, is that theoretically, the earth's gravity extends outward *forever*, though it gets weaker the farther out you go. Starting with Newton's law $F = ma$ and his Law of Gravitation, one can arrive† at the equation

$$v_{\text{escape}}^2 = 2\int_R^\infty \frac{GM}{r^2}\, dr,$$

where $R = 6.4 \cdot 10^6$ m is the radius of the earth, $G = 6.67 \cdot 10^{-11} \text{N} \cdot \text{m}^2/\text{kg}^2$ is the "gravitational constant," and $M = 6.0 \cdot 10^{24}$ kg is the mass of the earth. What is the earth's escape velocity?

Reading assignment for Wednesday, 2/11: On Wednesday we will finish Section 6.6 on improper integrals. Read all of pages 555 and 556 of your text (including Example 6.13) and the paragraph at the bottom of page 557. Skim the other examples of the comparison test, and reread any parts you'd like from the first half of the section. Prepare Exercise 39 and a reading question to be turned in in class.

Quote of the day:
"Our minds are finite, and yet even in these circumstances of finitude we are surrounded by possibilities that are infinite, and the purpose of life is to grasp as much as we can out of that infinitude."
—Alfred North Whitehead (1861 - 1947)

*Yeah, I don't really know how they came up with it, either. It's useful, though, and it shows where you'll use improper integrals in the future. Incidentally, we need $f(x)$ to be continuous on $[0, \infty)$.

†For the steps I've skipped, see the entry for "Escape Velocity" at wikipedia.org.