Merit Worksheet #9, 2/11/09

Improper integrals and the comparison test

1. For which values of k does the improper integral

$$\int_1^\infty \frac{1}{x^k}\,dx$$

converge?

2. For which values of k does the improper integral

$$\int_0^1 \frac{1}{x^k}\,dx$$

converge?

3. Do either of the integrals

$$\int_1^\infty \frac{1}{x^2}\,dx \qquad \text{and} \qquad \int_1^\infty \frac{1}{x^2+1}\,dx$$

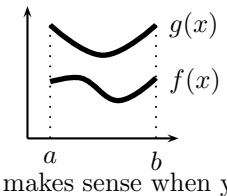
converge? What would you have to do to compute the value of $\displaystyle\int_1^\infty \frac{1}{x^2+x+1}\,dx$, if it converged? Would you have fun along the way?

As you read in your textbook, sometimes in order to test for the convergence of an improper integral we can't just integrate the function and take a limit, because finding the antiderivative is too hard (or we're just too lazy). In that case we *compare* the size of the integrand with the size of another (simpler) function.

Specifically, suppose $f(x)$ and $g(x)$ are never negative on the interval (a,∞).

$$\text{If } f(x) \le g(x) \text{ everywhere, then } \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

for any $b \ge a$.



4. (a) Explain why the inequality above makes sense when you look at the picture above.

   (b) So, looking at the inequality above, what can you say about $\displaystyle\int_a^\infty f(x)$ if $\displaystyle\int_a^\infty g(x)$ is finite?

   (c) What could be true about $\int_a^\infty f(x)$ in order to force $\int_a^\infty g(x)$ to diverge?

   These are important rules that allow us to decide whether or not an integral converges without actually having to try to compute the integral.

5. Use the facts you came up with in the previous problem to determine whether or not the following integrals converge (you do ***NOT*** have to find the value of the converging integrals).

   (a) $\displaystyle\int_{37}^\infty \frac{\sqrt{x}}{x-1}\,dx$ (b) $\displaystyle\int_2^\infty e^{-x^3}\,dx$ (c) $\displaystyle\int_{-\infty}^\infty \frac{|\sin 4x|}{1+x^2}\,dx$ (d) $\displaystyle\int_0^\infty \frac{1}{x^2+x+1}\,dx$

Consult now with your group and decide whether you'd like to try one of the following bonus problems, or whether you'd like to work as a group on the review exercises at the end of Chapter 6.

6. (Bonus) Show carefully that
$$\int_1^\infty \frac{1}{x^k}\,dx = \int_0^1 \frac{1}{x^{2-k}}\,dx,$$
where k is any constant. (Hint: make a u-substitution, AFTER writing the improper integrals as limits.) Show that your answers to Problems 1 and 2 make total sense in light of this problem.

7. (Bonus) Amazingly,
$$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}.$$
(Isn't that beautiful? You may see a proof of this in Calc III.) Use that information to compute
$$\int_{-\infty}^\infty e^{-kx^2}\,dx \qquad \text{and} \qquad \int_{-\infty}^\infty x^2 e^{-kx^2}\,dx.$$

Mock exam tomorrow night, Thursday, 2/12, 6-8 PM in 141 Altgeld Hall: The mock exam will be given during the first hour, and the answers will be gone over during the second hour. This is completely optional, but highly recommended. See you there!

Reading assignment for Friday, 2/13: On Friday we will review for the first midterm, which will be given during class on Monday and cover all of Chapter 6 (with the exception of Section 6.5). The first hour will involve an INTEGRATION BEE, where you will be competing as tables to see who are our class champs at integration. There will be prizes for the winners, so come ready to compete. For the second hour, we'll just be asking and answering questions, so please come to class having prepared questions to ask. Also, please try to have worked as many problems as possible from the review at the end of Chapter 6 in your text (including those 44 integrals!).

Mathematical joke of the day:

A somewhat advanced society has figured how to package basic knowledge in pill form. A student, needing some learning, goes to the pharmacy and asks what kind of knowledge pills are available. The pharmacist says "Here's a pill for English literature."

The student takes the pill and swallows it and has new knowledge about English literature! "What else do you have?" asks the student.

"Well, I have pills for art history, biology, and world history," replies the pharmacist. The student asks for these, and swallows them and has new knowledge about those subjects. Then the student asks, "Do you have a pill for math?"

"Wait just a moment," the pharmacist says. He goes back into the storeroom and brings back a whopper of a pill and plunks it on the counter.

"I have to take that huge pill for math?" inquires the student.

The pharmacist replies, "Well, you know math always was a little hard to swallow."