Merit Worksheet #10, 2/18/09

Differential equations

1. (a) What is a differential equation? (If you need a hint, look at a particular sentence in the middle of page 566 of your text.)
   (b) Which of the following are differential equations?

   $5x^2+11x+2=0 \qquad y=e^{x^2-x} \qquad y'(t)=4 \qquad x^2+y=7 \qquad x(y')^2-y^{(37)}\sin y=e^x \qquad xy'=1.$

2. (a) What does the solution to a differential equation look like? (See the sentence right after the one referred to in Problem 1.)
   (b) True or false? $y=x^3-3x^2+x+4$ is a solution to the differential equation $y''+y'+y=x^3+x-1$.
   (c) True or false? $x=3$ is a solution to the differential equation $(y')^2-3y'=0$.
   (d) How do you check to see if something is a solution to a differential equation?

3. Solve the following differential equations (make guesses, if you have to, and check to see if you're right):
   (a) $y'(t)=\dfrac{1}{\sqrt{1-t^2}}$
   (b) $y'(x)=0$
   (c) $y'(x)=y(x)$
   (d) $y''(x)=4\cos(x)$
   (e) $xy'=1$

4. The solution to a differential equation often involves an unknown constant. Sometimes a differential equation is given to us with an extra condition, called an *initial condition*, that allows us to solve for that constant. Solve the following problems (called *initial value problems*):

   (a) $y'(x)=x, \quad y(0)=5$ (b) $y'(t)=9y(t), \quad y(0)=2$

Solving differential equations can be very hard—although we can always check to see if a guess is correct, sometimes the solution is something you'd never guess. If you go on to take a class in differential equations, you'll learn a whole courseful of techniques for solving these problems. (And, having just learned a whole chapter full of techniques for finding integrals, doesn't that sound wonderful?)

There's one particular differential equation that shows up a lot in the sciences; it's the equation

$$y'(t)=ky(t), \tag{*}$$

where k is a constant. This equation is solved for you on pages 566 and 567 of your text.

5. Write down the solution to this differential equation (don't forget to include the constant A from your text!). What type of function is it (e.g., rational, polynomial, exponential, trigonometric, etc.)?

6. Suppose the function $y(t)$ in (*) represents a population. In your own words, in language you could have understood back in algebra, what is equation (*) saying about $y(t)$?

7. Suppose some quantity is increasing exponentially with growth rate r (i.e., it satisfies the differential equation $y'(t)=ry(t)$). Find how long it takes for the quantity to double.

8. Radioactive decay also obeys the differential equation (*). At 7 AM 100 grams of radioactive material are placed under observation. By 10 AM the amount has decayed to 90 grams.
   (a) Find an equation for the amount of radioactive material t hours after 7 AM.
   (b) What is the half-life of the substance, i.e., how long does it take for the half of the original material to decay?
9. It is reported that Prozac® has a half-life of 2 to 3 days but may be found in your system for several weeks after you stop taking it. What percentage of the original dosage would remain after 2 weeks if the half-life is 2 days? How much would remain if the half-life is 3 days?
10. (Bonus) Suppose you were told that $y' = x + 4y$. Now, solving this equation is not something we'll learn how to do in this course, but can you figure out a way to sketch what a solution might look like?

Preparation assignment for Friday, 2/20: On Friday we will cover Section 8.1. Read pages 612, 613, and 615 in your text, and skim the rest of the section. Prepare Exercises 1 and 5(a) to submit in class, along with a question you had while reading.

A little bit of academic rivalry:

Engineers think that their equations are an approximation to reality. Physicists think reality is an approximation to their equations. Mathematicians don't care.

A mathematician, a physicist, and an engineer are travelling through Scotland when they see a black sheep through the window of the train.

"Aha!" says the engineer. "I see that Scottish sheep are black."

"Hmm," says the physicist. "You mean that some Scottish sheep are black."

"No," says the mathematician, "All we know is that there is at least one sheep in Scotland, and that at least one side of that one sheep is black!"

The physicist and the engineer are in a hot air balloon. Soon they find themselves lost in a canyon somewhere. They yell out for help, "Hellllooooooo! Where are we?"

Fifteen minutes later, they hear an echoing voice: "Hellllooooooo! You're in a hot-air balloon!"

The physicist says, "That must have been a mathematician."

"Why do you say that?" the engineer asks.

"Because," replies the physicist, "His answer took a long time to get, it was absolutely correct, and it was utterly useless."