Merit Worksheet #11, 2/20/09

Infinite sequences

1. In your own words, what is a sequence? (See page 612, if you want, for the book's definition, but try to rephrase it in your own words.)

2. In the following, write out the terms $a_1, a_2, \ldots, a_6$ of the given sequence.

   (a) $a_n = \dfrac{3}{n+4}$ (b) $a_n = 1 + \dfrac{(-1)^n}{n!}$ (c) $a_n = \sqrt{n}$ (d) $a_n = \dfrac{(-1)^n n}{n+1}$

3. (a) What does the word "monotonous" mean in everyday English?

   (b) What does it mean for a sequence to be increasing? Decreasing? Monotonic? (See page 619 of your text.) Can you see a connection between the definition of "monotonic" and your answer to part (a)?

   (c) Which of the following sequences $\{a_n\}_{n=1}^{\infty}$ are monotonic?

   (i) $a_n = \dfrac{2n}{5n-3}$ (ii) $a_n = \dfrac{1+(-1)^n}{\sqrt{n}}$ (iii) $a_n = \tan n$ (iv) $1 - (2/3)^n$

4. (a) What does it mean for a sequence to be bounded? (See page 620 in your text for a formal definition, but try to put it in your own words.)

   (b) Which of the following sequences are bounded?

   (i) $a_n = \dfrac{2n}{5n-3}$ (ii) $a_n = [1+(-1)^n]n$ (iii) $a_n = \sqrt{\dfrac{2+\cos n}{n}}$ (iv) $1 - (3/2)^n$

5. Now let's talk about limits. Remember Calc I? What did it mean, back then, for

   $$\lim_{x\to a} f(x) = L?$$

   What does it mean when we say

   $$\lim_{x\to\infty} f(x) = L?$$

   What do you imagine it means, then, to say

   $$\lim_{n\to\infty} a_n = L?$$

6. Find the following limits, if they exist. (If a sequence's limit exists, we say that the sequence **converges** to that number.)

   (a) $\displaystyle\lim_{n\to\infty} \frac{3}{n+4}$ (b) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{(-1)^n}{n!}\right)$ (c) $\displaystyle\lim_{n\to\infty} \sqrt{n}$ (d) $\displaystyle\lim_{n\to\infty} \frac{(-1)^n n}{n+1}$

7. (a) Compare and contrast $\displaystyle\lim_{x\to\infty} \sin \pi x$ and $\displaystyle\lim_{n\to\infty} \sin \pi n$. Indicate the domains of the two functions and how they affect the limits.

   (b) What's the connection, if any, between

   $$\lim_{x\to\infty} f(x) \qquad \text{and} \qquad \lim_{n\to\infty} a_n?$$

8. Using limit rules (Theorem 1.1 on page 614) as appropriate, determine whether or not the sequence $\{a_n\}_{n=1}^{\infty}$ converges, and find its limit if it does converge:

(a) $a_n = \dfrac{1}{n^3}$ (b) $a_n = \dfrac{n}{n+1}$ (c) $a_n = 2 - \left(-\dfrac{1}{2}\right)^n$ (d) $a_n = n \sin\left(\dfrac{1}{n}\right)$

(e) $a_n = \dfrac{5n^3 - 1}{2n^3 + 1}$ (f) $a_n = (-1)^n \dfrac{n+4}{n+1}$ (g) $a_n = \dfrac{e^{2n} + 2}{e^n - 1}$ (h) $a_n = \cos(\pi n)$

9. (a) Does a monotonic sequence have to be convergent? If so, explain why; if not, give an example of a divergent monotonic sequence.
   (b) Does a convergent sequence have to be monotonic? If so, explain why; if not, give an example of a non-monotonic convergent sequence.
   (c) Does a bounded sequence have to be convergent? If so, explain why; if not, give an example of a divergent bounded sequence.
   (d) Does a convergent sequence have to be bounded? If so, explain why; if not, give an example of an unbounded convergent sequence.
   (e) Does a sequence which is both bounded and monotonic have to be convergent?

10. Based on the picture on page 614 and the text right before it, what does it mean for the sequence $\{a_n\}_{n=1}^{\infty}$ to converge to the limit L? (We're looking for an answer that involves ε here.) Write your answer below:

    Given some random ε, find N past which the terms of a_n differ from the sequence's limit by at most ε:

    (a) $a_n = \dfrac{1}{n^3}$ (b) $a_n = \dfrac{n}{n+1}$

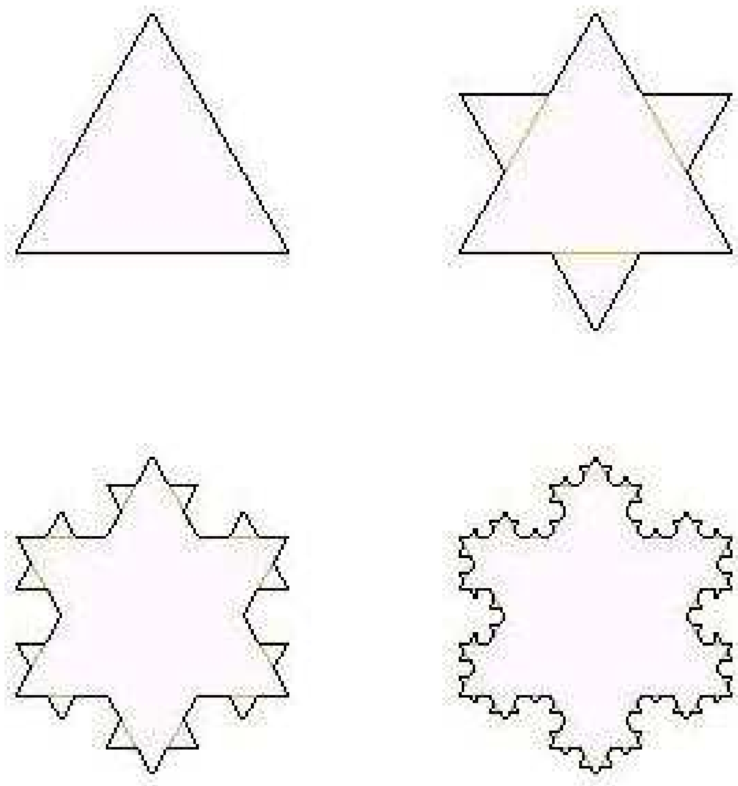
Preparation assignment for Monday, 2/23: On Monday we will begin our study of Section 8.2 in the text. Please read pages 626 through the end of Example 2.3 on page 629, and the two sentences immediately following the example. Then skim the rest of the section. For your preparation assignment, please turn in Exercise 26 (you might need a calculator for that one—send me an email if that's a problem for you) and, as always, a question you had while reading.

A triple feature of bonus materials!

Cool-online-encyclopedia-where-you-can-look-up-whatever-sequence-you-wish of the day:
http://www.research.att.com/~njas/sequences/
(My name is buried within the entry for Sequence A000009. Also, be sure to check out the puzzle page at http://www.research.att.com/~njas/sequences/Spuzzle.html.)

Quote of the day: "Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate." —Leonhard Euler (1707-1783)

Mathematical-oddity-about-sequences of the day:
Have you seen the Koch snowflake before? It's a *fractal* (a shape which is detailed—and often looks exactly the same—no matter how closely you zoom in on it) formed by starting with an equilateral triangle and then successively taking straight line segments and putting a "kink" in them. Putting it another way, it's the "limiting shape" for a sequence of *polygons* (neat, huh?). The first couple of iterations appear below; the snowflake is the curve they approach.



It's not surprising, perhaps, that the length of the snowflake is infinite. You can see that as you perform more iterations, adding more detail to the snowflake, the perimeters increase (you might say the perimeters form a monotone increasing sequence!). But here's what's remarkable: the same thing happens in unexpected places. The following is an excerpt from the Wikipedia article on the mathematician Lewis Fry Richardson (1881-1953):

"While studying the causes of war between two countries, Richardson decided to search for a relation between the probability of two countries going to war and the length of their common border. While collecting data, he realized that there was considerable variation in the various gazetted lengths of international borders. For example, that between Spain and Portugal was variously quoted as 987 or 1214 km while that between The Netherlands and Belgium as 380 or 449 km.

"As part of his research, Richardson investigated how the measured length of a border changes as the unit of measurement is changed. He published empirical statistics which led to a conjectured relationship. This research was quoted by mathematician Benot Mandelbrot in his 1967 paper 'How Long Is the Coast of Britain?'

"Suppose the coast of Britain is measured using a 200 km ruler, specifying that both ends of the ruler must touch the coast. Now cut the ruler in half and repeat the measurement, then repeat again. [See the picture below.]

“Notice that the smaller the ruler, the bigger the result. It might be supposed that these values would converge to a finite number representing the ‘true’ length of the coastline. However, Richardson demonstrated that the measured length of coastlines and other natural features appears to *increase without limit* as the unit of measurement is made smaller. Today this is known as the Richardson effect.

“Note that Richardson’s results do *not* mean that the coastline of Britain is actually infinitely long. This would require the ability to measure with infinitesimally small rulers, something which quantum physics says cannot be done, as there is a lower limit to the smallness of a measurement, the Planck length. What Richardson’s results do show is that natural geographic features, when considered over a wide range of scales, do not behave in the same way as the objects of Euclidean geometry.

“At the time, Richardson’s research was ignored by the scientific community. Today, it is seen as one element in the birth of the modern study of fractals.”

(Italics added; all images from wikipedia.org)

Cool, huh?