Merit Worksheet #13, 2/25/09

1. Match the items on the left with the appropriate items on the right.

sequence $\{a_k\}_{k=1}^{\infty}$	$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$
limit of a sequence	the sum of the first n terms, i.e., $a_1 + a_2 + \cdots + a_n$
series $\displaystyle\sum_{k=1}^{\infty} a_k$	a series of the form $\sum ar^k$
partial sum S_n	an infinite sum; the sum of a sequence
sum of a series	what the numbers in the sequence get closer and closer to
geometric series	the limit of the sequence of partial sums
$\sum ar^k$, when $\lvert r\rvert < 1$	when $\lvert r\rvert < 1$
harmonic series	a series where in each partial sum most middle terms cancel out
telescoping series	a list of numbers
when it is the series $\sum ar^k$ converges	$\dfrac{\text{first term}}{1-r}$

2. Examine the series

$$\sum_{k=0}^{\infty} (-1)^k.$$

   (a) Fill in the formula for the nth partial sum:

$$S_n = \begin{cases} \quad\quad , & \text{if } n \text{ is} \\ \\ \quad\quad , & \text{if } n \text{ is} \end{cases}$$

   (b) Does the series converge or diverge? How can you tell from the partial sums?

Special types of series, again

3. Which of the geometric series below converge? For those that converge, find the series' sum.

$$\sum_{k=0}^{\infty} (-1)^k \qquad \sum_{k=3}^{\infty} \frac{1}{2^k} \qquad \sum_{k=1}^{\infty} \frac{3^{2k-1}}{5^{k+1}} \qquad \sum_{k=1}^{\infty} \frac{1}{2} \qquad \sum_{k=2}^{\infty} \frac{5}{2^{k+1}}$$

4. True or false: Given *any* geometric series, there's a simple way (requiring fifteen seconds or less) to test whether or not the series converges.

5. What is the **harmonic series?** (See Example 2.7 in your text) Does the harmonic series converge or diverge?

6. Look again at Example 2.3 in your text. Why is that series called a **telescoping sum**? For the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)},$$

use partial fractions to write the summand in a different way. Then write the nth partial sum of the infinite series as a telescoping sum and find the sum of the series if it converges.

7. Find a fraction with integer numerator and denominator that equals

$$0.0514051405140514...,$$

i.e., that equals $0.\overline{0514}$. (As a bonus problem, which you should come back to *after* you finish the worksheet, use geometric series to prove that $0.99999... = 1$.)

8. Come back to this one later: Determine whether or not $\sum_{k=3}^{\infty} \ln\left(\frac{k}{k-1}\right)$ converges. (Hint: it's a telescoping series.)

9. As a group, come up with an explanation in your own words of what Theorem 2.3 in Section 8.2 means. Then evaluate

$$\sum_{k=0}^{\infty} \frac{7 \cdot 2^k - 3^k}{4^k}$$

The harmonic series and the kth-Term Test for Divergence

10. Look at the series below, which is called the **harmonic series.**

$$\sum_{k=1}^{\infty} \frac{1}{k}.$$

As opposed to geometric series or telescoping series, there is only one harmonic series (it's a single series, not a type of series).

(a) What is the limit of the individual terms (i.e., the summands) of the harmonic series?

(b) Explain why the following quantities are each bigger than 1/2:

$$\frac{1}{1}, \quad \frac{1}{2}, \quad \frac{1}{3}+\frac{1}{4}, \quad \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}, \quad \frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}$$

$$\frac{1}{17}+\frac{1}{18}+\frac{1}{19}+\frac{1}{20}+\frac{1}{21}+\frac{1}{22}+\frac{1}{23}+\frac{1}{24}+\frac{1}{25}+\frac{1}{26}+\frac{1}{27}+\frac{1}{28}+\frac{1}{29}+\frac{1}{30}+\frac{1}{31}+\frac{1}{32}.$$

(c) Based on part (b), how large can the partial sums of the harmonic series get?

(d) Does the harmonic series converge or diverge?

11. Determine whether each of the following statements are true or false. If a statement is true, give an explanation justifying it; if it is not, give an example of a series that shows that it is false. (Note: One of these statements is called the kth-Term Test for Divergence.)

(a) If $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ converges.

(b) If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(c) If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

(d) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\lim_{k\to\infty} a_k \neq 0$.

12. Give another reason why the series in Problem 2 diverges.

13. Which of the following series are *guaranteed* to diverge by the kth-Term Test for Divergence?

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{2k}{k+3}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k+3}{k^2+3k}$ (c) $\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ (e) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n}{n+4}$ (f) $\displaystyle\sum_{k=0}^{\infty} ar^k$, for $|r| > 1$.

Preparation assignment for Friday, 2/27:

Next time we will learn about the integral test and the comparison tests in Section 8.3. The best way to learn about these tests is to see examples and work problems yourself. For that reason, the reading assignment will be a bit longer. Please skim the argument on pages 636 and 637, and then reread Theorem 3.1 and Example 3.1, followed by the boxed statement on page 639. Read Theorem 3.3 on page 641 and the paragraph directly above and below it. Then read Examples 3.5, 3.6, and 3.7, and the paragraph after Example 3.7. Read Theorem 3.4 and Examples 3.8 and 3.9. For your problem, test the convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{4}{4k-2}$ in three ways—once using the integral test, once by using the comparison test (comparing it to the harmonic series), and once more by using the limit comparison test (again using the harmonic series). And again, don't forget to write down a reading question!

Quote of the day:
"If you disregard the very simplest cases, there is in all of mathematics not a single infinite series whose sum has been rigorously determined. In other words, the most important parts of mathematics stand without a foundation." — Niels H. Abel (1802 - 1829), a great Norwegian mathematician.

Some series to chew on as we go through Chapter 8

Sections 8.2 through 8.5 are all about testing whether or not a series converges. As you had to do with integrals, on the next midterm you'll be given a series and told to decide whether or not it converges, but it'll be up to you to choose an appropriate test (and, just as with integrals, there may be several ways to correctly do the problem). As we go through Chapter 8, please refer back to this list every day and see on which series the tests you learn work, and on which they don't.

Determine if the following series converge or not:

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)}$

(b) $\displaystyle\sum_{k=0}^{\infty} (-2)^k$

(c) $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k+1}$

(d) $\displaystyle\sum_{k=0}^{\infty} \sin k$

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^3+1}$

(f) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{5}{2}\right)^{-k}$

(g) $\displaystyle\sum_{k=0}^{\infty} \frac{\tan^{-1} k}{1+k^2}$

(h) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln \sqrt{k}}{k}$

(j) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$

(k) $\displaystyle\sum_{k=1}^{\infty} \frac{k+\sqrt{k}}{2k^3-\sqrt{k}}$

(l) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$

(m) $\displaystyle\sum_{k=1}^{\infty} \sin(\pi/k)$

(n) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\cos \pi k}{k}$

(o) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2+1}$

(p) $\displaystyle\sum_{k=0}^{\infty} \sin\left(\frac{k\pi}{4}\right)$

(q) $\displaystyle\sum_{k=2}^{\infty} (-1)^k \frac{1}{k \ln k}$

(r) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^k$

(s) $\displaystyle\sum_{k=2}^{\infty} \frac{3}{10^k}$

(t) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^2-k}$

Though you won't be expected to find the sum of most series (because it's beyond what mathematicians can do exactly—hence today's Quote of the Day), you ***should*** be able to find the sums of the series in parts (a), (f), (s), and (t).