Merit Worksheet #14, 2/27/09

Ten-minute review: the harmonic series and the kth-Term Test for Divergence

1. Here's a quick review of an important concept: What's the series shown below called?

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

   (a) Do the individual terms of the series converge? If not, why not, and if so, what to?

   (b) Does the series as a whole converge? (Does the answer surprise you? We'll prove this later in the worksheet.)

2. Give a conceptual explanation of why each of the following series diverges:

$$\text{(a)} \quad \sum_{k=0}^{\infty} 2 \qquad \text{(b)} \quad \sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right) \qquad \text{(c)} \quad \sum_{k=0}^{\infty} (-1)^k$$

3. Which of the following statements are always true?

   (i) If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

   (ii) If $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ converges.

   (iii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\lim_{k\to\infty} a_k \neq 0$.

   (iv) If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

   Your textbook authors call the first *true* statement above the kth-Term Test for Divergence (circle it!). Give an example of a series that shows that the two false statements really are false.*

4. Which of the following series are *guaranteed* to diverge by the kth-Term Test for Divergence?

$$\text{(a)} \quad \sum_{k=0}^{\infty} \frac{2k}{k+3} \qquad \text{(b)} \quad \sum_{k=1}^{\infty} \frac{k+3}{k^2+3k} \qquad \text{(c)} \quad \sum_{k=1}^{\infty} n \sin\left(\frac{1}{n}\right)$$

$$\text{(d)} \quad \sum_{n=1}^{\infty} \frac{n}{n+1} \qquad \text{(e)} \quad \sum_{n=0}^{\infty} (-1)^n \frac{n}{n+4} \qquad \text{(f)} \quad \sum_{k=0}^{\infty} ar^k, \text{ for } |r| > 1.$$

The integral test

5. The **integral test** says the following:

   If $a_k = f(k)$ for all k, where the function $f(x)$ is
   - **continuous**,
   - **decreasing**, and
   - **never negative when** $x \geq 1$,

   then $\int_1^{\infty} f(x)\,dx$ and $\sum_{k=1}^{\infty} a_k$ either both converge or both diverge.

*The second true statement is just another way of saying the first true statement—logically, it's called the *contrapositive* of the statement.

Explain in your own words why this theorem intuitively makes sense (***as a group,*** look at the figures on page 637 of your text and explain what they have to do with the theorem).

6. Use the integral test to show that the harmonic series diverges.

7. See if either the kth-term test for divergence or the integral test apply to any of the series below, and if so, decide whether the series converges or diverges:

$$\text{(a)} \quad \sum_{k=1}^{\infty} \frac{4}{\sqrt[3]{k}} \qquad \text{(b)} \quad \sum_{k=2}^{\infty} \frac{1}{k \ln k} \qquad \text{(c)} \quad \sum_{k=4}^{\infty} \frac{2k}{k^2+1}$$

In each case, which test was more useful? Why?

8. Use the integral test (and not your textbook) to fill in the blank below:

> A series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p-**series**.
>
> A p-series written this way will converge **exactly when** p ____________________

(Hint: when you set up the integral you should be able right away to use a fact you memorized for the last midterm.)

The comparison tests

9. Decide whether the series below converge or diverge by using the comparison test:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{1}{n^2+1} \qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

10. Decide whether the series below converge or diverge by using the limit comparison test:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}} \qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{4n+1}{3n^3-n^2-1}$$

Preparation assignment for Monday, 3/2:

On Monday we will continue our discussion of the comparison tests from today's worksheet. For next time, review Examples 3.5 through 3.9 in Section 8.3, *especially* Example 3.7. Write up answers to ***Writing Exercises*** 1 and 4 at the end of Section 8.3, and write down a question you have on series—this time it can be about anything having to do with what we've covered so far.

Stupid math joke of the day

A mathematician organizes a lottery in which the prize is an infinite amount of money. When the winning ticket is drawn, and the jubilant winner comes to claim his prize, the mathematician explains the mode of payment: "1 dollar now, 1/2 dollar next week, 1/3 dollar the week after that..."