Merit Worksheet #15, 3/2/09

The comparison tests

1. Fill in the blank below, either from memory, your textbook, or the integral test:

   A series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p**-series**.

   A p-series written this way will converge **exactly when** p ____________________

2. Decide whether the series below converge or diverge by using the comparison test:

   (a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}}$

3. Decide whether the series below converge or diverge by using the limit comparison test:

   (a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$ (b) $\sum_{n=1}^{\infty} \frac{4n+1}{3n^3-n^2-1}$

4. Determine whether or not the following series converge. (Use any method you wish.)

   (a) $\sum_{k=1}^{\infty} \frac{1}{k+\sqrt{k}}$ (b) $\sum_{k=1}^{\infty} \frac{1+\sqrt[3]{k}}{k}$ (c) $\sum_{k=1}^{\infty} \frac{1}{\ln k}$ (d) $\sum_{k=4}^{\infty} \frac{1}{k\sqrt{k-3}}$

   (e) $\sum_{k=1}^{\infty} \frac{3^k+1}{5e^k+k}$ (f) $\sum_{k=1}^{\infty} \frac{1}{(\ln k)^k}$ (g) $\sum_{k=1}^{\infty} \frac{5+\sqrt{k}}{1+k}$ (h) $\sum_{k=0}^{\infty} \frac{1}{1+k^2}$

   (i) $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ (j) $\sum_{k=1}^{\infty} \frac{2k-1}{k^2}$ (k) $\sum_{k=1}^{\infty} \frac{n}{e^n}$

5. Determine whether the following quantities converge:

   (a) $\sum_{k=1}^{\infty} \frac{k^2 e^{-k}}{1+k^2}$ (b) $\int_1^{\infty} \frac{x^2 e^{-x}}{1+x^2}\, dx.$

   What connections do you see among (i) the comparison test for series, (ii) the comparison test for improper integrals, and (iii) the integral test for series?

6. Suppose you know that the series $\sum_{k=1}^{\infty} a_k$ converges. How can you tell, without even needing to know what the terms a_n are, that the series $\sum_{k=1}^{\infty} a_k^2$ is going to converge as well?

7. (Bonus) Find two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that

   $$\lim_{n\to\infty} \frac{b_n}{a_n} = 0, \qquad \sum_{n=1}^{\infty} b_n \text{ converges}, \qquad \text{and } \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

   Explain why this does not contradict the Limit Comparison Test.

Review problems

For each of the following series, determine whether or not the series converges. If it does, find its sum; if it doesn't, explain how you know that it doesn't.

$$\text{(A)} \quad \sum_{n=1}^{\infty} \sin^n 1 \qquad\qquad \text{(B)} \quad \sum_{k=2}^{\infty} 3\left(-\frac{9}{10}\right)^k$$

Preparation assignment for Wednesday, 3/4:

We will cover Section 8.4 on Wednesday. Please come to class having read from the beginning of the section up through the statement of Theorem 4.1 (the Alternating Series Test), as well as Examples 4.2, 4.3, and 4.4. Then read Theorem 4.2 and Example 4.5. Prepare Writing Exercise 4 and (regular) Exercise 1 to turn in during class, along with a reading question.

A moment in history

Did you know that the content of calculus courses has changed over time? You should consider yourself privileged, given what Henry Scheffé of Oregon State College had to say in 1941:

> "Elementary calculus texts do not commonly include the following:
>
> "SERIES COMPARISON TEST. *Let*
>
> $$u_1 + u_2 + u_3 + \dots \tag{1}$$
>
> *and*
>
> $$v_1 + v_2 + v_3 + \dots \tag{2}$$
>
> *be two series of positive terms. Then if*
>
> $$\lim_{n\to\infty} u_n/v_n = 1,$$
>
> *the series converge or diverge together; that is, if* (2) *converges, so also does* (1)*; and if* (2) *diverges, so does* (1)."

Do you recognize that test? (In fact, you've learned a more powerful version of this test.) Scheffé went on to say,

> "[S]tudents seem to find the familiar calculation of a limit easier than the manipulation of inequalities.* If facility in such manipulation is an objective of the course, our test may not recommend itself."

If you'd like me to give you more practice in algebraic manipulation, let me know. But don't you see? You're learning things in here that even your grandparents may not have learned! Isn't it a wonderful time to be alive and taking calculus?

*Here he's talking about the ordinary comparison test.