Merit Worksheet #17, 3/6/09

Quick, 5-minute review

Seen on a quiz: Determine whether or not $\sum_{k=0}^{\infty} \frac{k}{k^2+k+1}$ converges.

What's wrong with the reasoning in the following answer?

$$\text{“}\lim_{k\to\infty} \frac{k}{k^2+k+1)} = \lim_{k\to\infty} \frac{1}{2k+1} = 0, \text{ so } \sum_{k=0}^{\infty} \frac{k}{k^2+k+1} \text{ converges.”}$$

(How could you show the convergence or divergence instead?)

Absolute convergence

1. What does it mean for a series to converge absolutely? Conditionally? (See pg. 656 of your text.)
2. Which of the following series converge absolutely? Conditionally? Not at all?

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \qquad \sum_{k=1}^{\infty} \frac{1}{k} \qquad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

3. Now look again at Theorem 5.1 on page 657. What does it mean, intuitively? (Think of each term in the series as a step along the number line, and think about what happens if either $\sum |a_k|$ or $\sum a_k$ is infinite.)
4. You may be asking yourself (or you may have already asked me) why we care about absolute convergence. Answer the following.
   (a) Name three convergence tests you've learned so far that *only* work if the series terms are all positive?
   (b) Name the only type of series we've talked about so far (last Wednesday) which has both positive and negative terms and which we have a convergence test for?
   (c) Now, designate someone in your group to read Example 5.3 (pages 657-658) *out loud* (with feeling, if you want). Pay particularly close attention to the first two sentences of the solution. So...why do we care about absolute convergence?

The two tests in Section 8.5—the ratio test and the root test—tell us whether or not a series converges *absolutely.*

Geometric series again, really quickly

5. Under what circumstance does the geometric series $\sum_{k=0}^{\infty} ar^k$ converge?

The ratio test

6. Review the ratio test, presented on page 658 of your text. Then use the ratio test to decide whether or not the following series converge:

$$\text{(a)} \quad \sum_{k=1}^{\infty} \frac{(-1)^k k}{3^k} \qquad \text{(b)} \quad \sum_{k=1}^{\infty} \frac{10^n}{n!} \qquad \text{(c)} \quad \sum_{n=1}^{\infty} \frac{e^n}{n}$$

7. For each of the geometric series below, what's the ratio a_{k+1}/a_k?

$$\text{(a)} \quad \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \qquad \text{(b)} \quad \sum_{k=1}^{\infty} 10 \cdot 3^k \qquad \text{(c)} \quad \sum_{k=0}^{\infty} ar^k$$

8. How might the ratio test show you how much a series behaves like a geometric series?

9. What does the ratio test tell you about each of the following series?

$$\text{(a)} \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \qquad \text{(b)} \quad \sum_{k=1}^{\infty} \frac{1}{k^2+1} \qquad \text{(c)} \quad \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$$

Looking ahead a bit (to the Root Test)

9. Quick! As a bit of preparation for the next problems, find the following:

$$\text{(a)} \quad \lim_{k\to\infty} 2^{1/k} \qquad \text{(b)} \quad \lim_{k\to\infty} k^{1/k} \qquad \text{(c)} \quad \lim_{k\to\infty} \left(k^2\right)^{1/k}$$

A bonus problem

10. The *Fibonacci sequence* is the sequence F_k where $F_1 = 1$, $F_2 = 1$, and $F_k = F_{k-2} + F_{k-1}$ for every $k \geq 3$. (In words, each term is the sum of the two previous terms.) The first terms of the Fibonacci sequence are listed below:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

The Fibonacci sequence has many remarkable properties. One is that it behaves almost like a geometric sequence. If you look at the ratios

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \frac{144}{89}, \frac{233}{144}, \frac{377}{233}, \frac{610}{377}, \frac{987}{610}, \ldots,$$

you'll see that they seem to approach a single number, approximately 1.618.

(a) Suppose the limit F_{k+1}/F_k of consecutive terms is L. Working with the equation

$$F_k = F_{k-2} + F_{k-1}, \qquad \text{we get} \qquad \frac{F_k}{F_{k-1}} = \frac{F_{k-2}}{F_{k-1}} + 1.$$

What should the exact value of L be?

(b) Use the ratio test to determine whether or not the series $\sum_{k=1}^{\infty} \frac{1}{F_k}$ converges.

Preparation for next time, and a stupid math joke

Read the section on the Root Test (page 661) and the “Summary of Convergence Tests” that follows after it. Then, using the Root Test and the fact that $\lim_{k\to\infty} k^{1/k} = 1$, answer Exercise 11 on page 663. Turn that problem and a reading question in on Monday.

Two mathematicians are studying a convergent series.
The first one says, “Do you realize that the series converges even when all the terms are made positive?”
The second one gasps, “Are you sure?”
“Absolutely!”