Merit Worksheet #18, 3/9/09

Geometric series yet again, even more quickly

1. Under what circumstance does the geometric series $\sum_{k=0}^{\infty} ar^k$ converge?

The root test

2. Quick! As a bit of preparation for the next problems, find the following (and write out <u>all</u> the steps of each limit on the board):

   (a) $\lim_{k\to\infty} 2^{1/k}$ (b) $\lim_{k\to\infty} k^{1/k}$ (c) $\lim_{k\to\infty} \left(k^2\right)^{1/k}$

3. Review the Root Test, stated on page 661 of your text. Then use the root test to determine whether or not the series below converge:

   (a) $\sum_{k=1}^{\infty} \frac{(-1)^k k}{3^k}$ (b) $\sum_{k=1}^{\infty} \left(\frac{3k-1}{4k+6}\right)^k$ (c) $\sum_{n=1}^{\infty} \frac{e^n}{n}$

4. For each of the geometric series below, what's the limit $\lim_{k\to\infty} (a_k)^{1/k}$?

   (a) $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ (b) $\sum_{k=1}^{\infty} 10 \cdot 3^k$ (c) $\sum_{k=0}^{\infty} ar^k$

5. (a) How might the Root Test show you how much a series behaves like a geometric series?
   (b) What other convergence test have we looked at recently that looked at how "geometric" a series is? In what ways are these two convergence tests similar to the geometric series condition for convergence?

6. What does the root test tell you about each of the following series?

   (a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ (b) $\sum_{k=1}^{\infty} \frac{1}{n^2+1}$ (c) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

Most definitely *NOT* a bonus problem. Do not skip this next one.

7. This Saturday is March 14, known throughout the world as π Day (can you guess why?). When the event falls on a school day (and maybe even this year), the math club here on campus usually meets on the Quad at 1:59 PM (again, can you guess why?) to have pie-eating contests and recitation contests. That's right; people compete to see who can recite the most digits of π. I once had a friend that knew 80 digits by heart. I myself only have 12 down. But here's a question which may be bothering you. ***How do we know what π is supposed to be, exactly?*** (I promise you, it isn't computed by drawing a really large circle and measuring its circumference and diameter.)

   One (bad) way of calculating π is to use the *Leibnitz formula*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1}.$$

(a) Show that this series converges.

(b) Suppose we took the 100th partial sum of this series. How far away from $\pi/4$ could that answer be?

(c) Suppose we wanted to compute $\pi/4$ to within $1/1000$ of its true value. Which partial sums would be good enough to achieve this?

Or this next one, either!

8. The *golden ratio* is the quantity $\phi = (1+\sqrt{5})/2$. You may have heard of it before; the Greeks used it for architecture all the time, and it shows up surprisingly often in nature. It's also closely related to the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

where each term is the sum of the two terms below it. (If you did the bonus problem last time, you've seen how it's related.)

Well, let's do the following, let's look at different powers of ϕ, and see what numbers they round to. For example, $\phi^1 = 1.618...$, which I'll fudge a little and round down to 1. If I were to round ϕ^2, I'd get 3, and ϕ^3 rounds to 4.

(a) Using a calculator, fill in the following blanks on the "roundings":

$$\underbrace{1}_{\phi^1}, \underbrace{3}_{\phi^2}, \underbrace{4}_{\phi^3}, \underbrace{\quad}_{\phi^4}, \underbrace{\quad}_{\phi^5}, \underbrace{\quad}_{\phi^6}, \underbrace{\quad}_{\phi^7}, \underbrace{\quad}_{\phi^8}, \underbrace{\quad}_{\phi^9}, \underbrace{\quad}_{\phi^{10}}, \ldots$$

(b) The sequence above is known as the *Lucas sequence*, which we'll label as $\{L_k\}_{k=1}^{\infty}$. What Fibonacci-like pattern do you notice in its terms?

(c) Use the Root Test to determine whether or not the series

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{4} + \frac{1}{L_4} + \frac{1}{L_5} + \ldots,$$

(in other words, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{L_k}$) converges.

Preparation for next time

Next time we'll review all the series convergence tests. **There will be nothing to turn in for next time,** but in order to be ready, you should work all the problems on this worksheet, all of the series at the end of Worksheet #13, and some exercises from Section 8.5 in your text.

Watching *High School Musical* makes you smarter!

About a year ago I attended a math conference at Illinois State University. One of my favorite talks was by Professor Bruce C. Berndt (from our very own department—take a class from him, if you can) about the famous Indian mathematician Ramanujan and series he discovered which converge to $1/pi$. Two of these series can be written as

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} \frac{(6k+1)}{4^k} \left[\frac{(1/2)(1/2+1)(1/2+2)\ldots(1/2+k-1)}{k!} \right]^3$$

and

$$\frac{16}{\pi} = \sum_{k=0}^{\infty} \frac{(42k+5)}{2^{6k}} \left[\frac{(1/2)(1/2+1)(1/2+2)\ldots(1/2+k-1)}{k!} \right]^3 .$$

Using a bit of math notation not commonly taught in high school, and changing the k's to n's, we can write that last one more concisely as

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(42n+5)\left(\frac{1}{2}\right)_n^3}{64^n (n!)^3}.$$

Professor Berndt pointed out that if you've watched Disney's *High School Musical*, then you've seen this formula before! There's a scene where Gabriella Montez corrects a formula her teacher's written on the board, prompting her to change an 8 into a 16 in the formula above. If you'd like to see for yourself, you can visit a blog which points this out (and has some screenshots from the movie) at `http://unimodular.net/blog/?p=116`. I have to agree with Sharpay's open-mouthed reaction to Gabriella's catch, but I have to wonder (a) why the math class is being taught in a chemistry lab, and (b) why the high school class is covering graduate-level analytic number theory.

Amazing, right? See why I go to math conferences?