

Merit Worksheet #19, 3/11/09

HERE ARE THE RULES: Work the problems below in the randomly-generated order listed on your group's participation sheet. You may NOT move on to the next problem, even individually, until everyone in your group has gotten an answer to the problem you're on, and understands that answer—so *talk* to each other, ask each other questions, and answer each other's questions. When you're assigned to work on Problem 25, just test one or two of the series there before moving on to the next problem.

This worksheet should be a really good review for Monday's midterm, so even after class is over, work on it until you've covered it all.

One giant review of sequences and series

1. **Sequences.** What is a sequence? What does it mean for a sequence to have a limit?
2. Using limit rules, determine whether or not the sequence $\{a_n\}_{n=1}^{\infty}$ converges, and find its limit if it does converge:

$$(a) \quad a_n = \frac{4n}{10 - 3n} \quad (b) \quad a_n = \frac{\sin n}{\cos n} \quad (c) \quad a_n = 2 - \left(-\frac{1}{2}\right)^n \quad (d) \quad a_n = n \sin\left(\frac{1}{n}\right)$$

3. (a) What does it mean for a sequence to be monotonic? Give an example of a monotonic sequence.
(b) What does it mean for a sequence to be bounded? Give an example of a bounded sequence.
(c) True or false: a convergent sequence must be monotonic.

4. **Geometric series.** Which of the following are geometric series?

$$\sum_{k=0}^{\infty} \frac{1}{3^k} \quad \sum_{k=4}^{\infty} k^2 \quad \sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n+1}} \quad \sum_{k=1}^{\infty} 1 \quad \sum_{n=0}^{\infty} \frac{2^n - 3^n}{4^n} \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{k} \quad \sum_{n=2}^{\infty} \frac{10}{3 \cdot 7^n} \quad \sum_{n=0}^{\infty} \pi^n$$

5. For each of the geometric series in the last problem, state whether the series is convergent or divergent. If it is convergent, give its value.
6. **The k th-Term Test for divergence.** True or false?

(a) In the series $\sum_{k=1}^{\infty} \frac{k}{10k + 17}$, since $\lim_{k \rightarrow \infty} k/(10k + 17) = 1/10$, we know the series converges to $1/10$.

(b) In the series $\sum_{k=1}^{\infty} \frac{2k}{\sqrt{4k^2 + 3}}$, since $\lim_{k \rightarrow \infty} 2k/\sqrt{4k^2 + 3} = 1$, we know the series diverges.

(c) In the series $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$, since $\lim_{k \rightarrow \infty} 1/(k \ln k) = 0$, we know the series converges.

(d) In the series $\sum_{k=2}^{\infty} \frac{1}{2^k}$, since $\lim_{k \rightarrow \infty} 1/2^k = 0$, we know the series converges to 0.

(e) If the series $\sum a_k$ converges, then the sequence a_k has to converge.

(f) If the sequence a_k converges, then the series $\sum a_k$ has to converge.

7. **Telescoping series.** Write the n th partial sum of the infinite series as a telescoping sum and thereby find the sum of the series if it converges:

$$\sum_{k=2}^{\infty} \frac{1}{(k-1)k}$$

8. **p -series.** What is a p -series? When does it converge? (And, for that extra feeling of mastery, can you prove the convergence rule you just stated, using one or another of the convergence tests?)
9. **The Integral Test.** Under what conditions *can* you apply the Integral Test to a series? When will it be *useful* to apply the Integral Test?
10. Use the Integral Test to determine whether the following series converge:

$$(a) \sum_{k=1}^{\infty} \frac{k}{(4k^2 + 5)^{3/2}} \quad (b) \sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}} \quad (c) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$$

11. True or false? If $a_k = f(k)$ and the Integral Test applies to the series $\sum_{k=1}^{\infty} a_k$, then $\sum_{k=1}^{\infty} a_k = \int_1^{\infty} f(x) dx$. Why is your answer correct?

12. **The Comparison Test.** Decide whether the series below converge or diverge by using the Comparison Test:

$$(a) \sum_{k=1}^{\infty} \frac{1}{k^2 + 1} \quad (b) \sum_{k=2}^{\infty} \frac{1}{\sqrt{k+1}}$$

13. **The Limit Comparison Test.** Decide whether the series below converge or diverge by using the Limit Comparison Test:

$$(a) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2 + k}} \quad (b) \sum_{k=1}^{\infty} \frac{4k + 1}{3k^3 - k^2 - 1}$$

14. **Alternating series.** Decide whether or not the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{3k^2 + 2}$ converges or diverges.

15. Does the series in the last problem converge absolutely, converge conditionally, or diverge?

16. **The Ratio Test.** Decide whether the series below converge or diverge by using the Ratio Test:

$$(a) \sum_{k=1}^{\infty} \frac{k}{2^k} \quad (b) \sum_{k=1}^{\infty} \frac{10^k}{k!}$$

17. Explain why the Ratio Test won't work on the following series:

$$(a) \sum_{k=1}^{\infty} \frac{1}{k^2 + 1} \quad (b) \sum_{k=1}^{\infty} \frac{1}{\ln k}$$

18. **The Root Test.** Decide whether the series below converge or diverge by using the Root Test:

$$(a) \sum_{k=1}^{\infty} 2^k \quad (b) \sum_{k=1}^{\infty} k e^{-k}$$

19. Explain why the Root Test won't work on the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

20. True or false: The Ratio Test and Root Test *never* work on a conditionally convergent series.

21. Suppose you're examining the series $\sum_{k=1}^{\infty} \frac{p(k)}{q(k)}$, where $p(k)$ is a polynomial of degree ℓ and $q(k)$ is a polynomial of degree m . Use one of the convergence tests, plus what you know about p -series, to come up with a simple rule for deciding when the given series converges.

More on alternating series

22. On a number line, sketch an example of where the partial sums S_n might fall for a series which satisfies the Alternating Series Test.
23. Say you have a series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ which satisfies the conditions of the Alternating Series Test. What's the farthest that the partial sum $\sum_{k=1}^n (-1)^{k+1} a_k$ could be away from the sum of the series?
24. Suppose you wanted to find a partial sum of the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ that was guaranteed to be within 1/100 of the series' sum. How many terms would your partial sum need?

All of the tests, combined.

25. Decide whether the following series converge or diverge by using any of the tests we've learned. Feel free to skip around and work whichever series you want. For an extra feeling of satisfaction, and the admiration of your classmates, try to work at least one or two series using *each* of the methods above. Super-bonus points if you can use two different ways to decide whether or not the same series converges.

$\sum_{k=1}^{\infty} \frac{\sin^2 k}{3^k}$	$\sum_{k=1}^{\infty} \frac{1}{k + \sqrt{k}}$
$\sum_{k=1}^{\infty} (-1)^k \frac{2}{k^2}$	$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{k+1}$
$\sum_{k=1}^{\infty} \frac{1 + \sqrt[3]{k}}{k}$	$\sum_{k=1}^{\infty} \frac{1}{2^k \sqrt{k}}$
$\sum_{k=1}^{\infty} \frac{1}{k^k}$	$\sum_{k=1}^{\infty} \frac{1}{[3 + (-1)^k]^k}$
$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{2^k}$	$\sum_{k=1}^{\infty} (-1)^k \frac{4^k}{k!}$
$\sum_{k=1}^{\infty} k^3 \left(\frac{3}{4}\right)^k$	$\sum_{k=1}^{\infty} \frac{k}{2k+1}$
$\sum_{k=1}^{\infty} \frac{1}{k^2 - 3}$	$\sum_{k=1}^{\infty} \frac{1}{\sqrt{2k^3 - k}}$
$\sum_{k=1}^{\infty} \frac{3^k + 1}{5e^k + k}$	$\sum_{k=1}^{\infty} \frac{1}{4k - 1}$
$\sum_{k=1}^{\infty} \frac{5 + \sqrt{k}}{1 + k}$	$\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k - 1)}$
$\sum_{k=1}^{\infty} \frac{k^3}{k!}$	$\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)(k+3)(k+5)}$$

$$\sum_{k=1}^{\infty} \frac{k^k}{k!}$$

For next time

Again, there will be nothing to turn in next time. There will also be no new homework assigned today. Spend your time going over this worksheet and preparing for the test. Remember, the mock exam is **tomorrow (Thursday) from 6 to 8 PM in 141 Altgeld Hall, and the exam is on Monday from 11 to 12:50 (our normal class time) in 164 Noyes Lab.**

Two quotes of the day from celebrated mathematicians

“Every mathematician worthy of the name has experienced ... the state of lucid exaltation in which one thought succeeds another as if miraculously... this feeling may last for hours at a time, even for days. Once you have experienced it, you are eager to repeat it but unable to do it at will, unless perhaps by dogged work...” — André Weil (1906 – 1998)

May you experience this miraculous thought-succession as you study for your test! And may you not experience the following:

“A formal manipulator in mathematics often experiences the discomfoting feeling that his pencil surpasses him in intelligence.” — Howard W. Eves (1911 – 2004)