Merit Worksheet #22, 3/20/09

Power series

1. Look at the series $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$.

   (a) Does the series converge? If so, then what to?

   (b) Write out the first five partial sums and simplify them (use a calculator, if you'd like).

   (c) What obvious connection do you see between your answers to (a) and (b)?

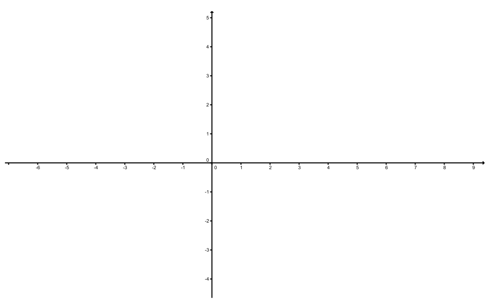
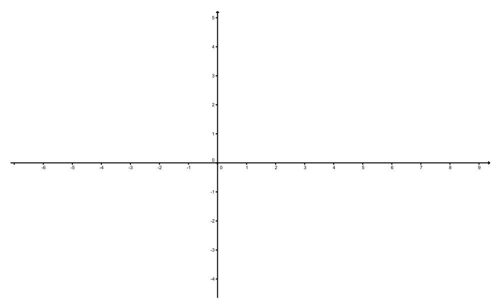
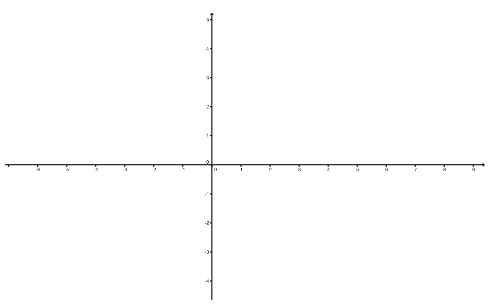
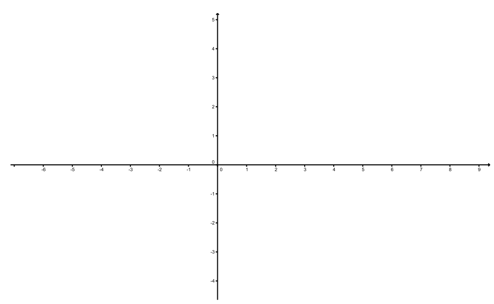
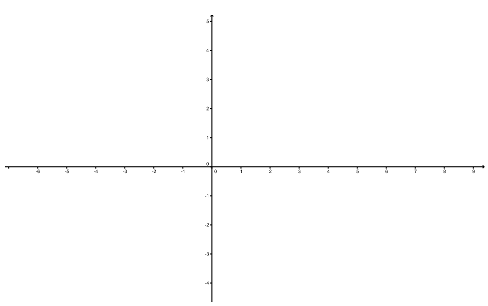
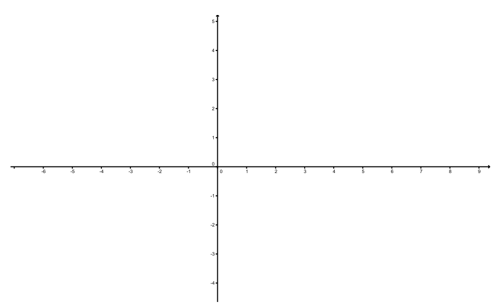
2. Now let's look at the series $\sum_{k=0}^{\infty} x^k$.

   (a) What's the interval of convergence for this series?

   (b) When this series converges, what does it equal?

   (c) Write out the first five partial sums of this series (S_0 through S_4).

   (d) On each of the axes below, sketch the graph of your answer to part (b). Then sketch your answers to part (c), one partial sum per set of axes, on the first five sets of axes. On the last set of axes, sketch S_{100} (as seen in class).

(e) What are the graphs doing as we take larger partial sums? How are they behaving differently *inside* the interval of convergence versus *outside* the interval of convergence?

3. Now let's look at the series from the quiz problem: $\sum_{k=1}^{\infty} \frac{(x-1)^k}{k^2}$.

   (a) What's the interval of convergence for this series?

   (b) What are the first four partial sums of the series? Find someone with a graphing calculator and have them graph each of these partial sums. Sketch the graphs on the chalkboard.

   (c) What do you think will happen to the graphs as we take more and more partial sums?

   (d) Do you know what curve the graphs of the partial sums seems to approach? (And does that bother you?)

In general, any old power series $\sum a_k(x-c)^k$ is going to converge to some (usually unknown) function inside the interval of convergence. If you get bored over spring break, try making up power series and seeing what graph the partial sums seem to settle on.

Now, we *can* figure out what some power series converge to, as the next few problems will show.

4. As you read in preparation for today, if we differentiate (or integrate) each term of a series, it becomes a power series which equals the derivative (or integral) of the function the original function equals. Based on what you know about $\sum x^k$, what functions do the series

$$\sum_{k=0}^{\infty} kx^{k-1} \qquad \text{and} \qquad \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

   equal?

5. Use the power series $\sum_{k=0}^{\infty} (-1)^k x^k$ to find power series representations of $\frac{1}{(1+x)^2}$, $\frac{1}{1+x^2}$, and $\tan^{-1} x$.

6. Use your answer to Problem 4 to find the sum of the following series:

$$\text{(a)} \quad \sum_{k=0}^{\infty} \frac{k}{2^{k-1}} \qquad\qquad \text{(b)} \quad \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)3^{k+1}}$$

The interval of convergence

You've seen that if you have a power series equal to a function $f(x)$, then to get a power series equal to $f'(x)$, all you need to do is differentiate each term of the original series. The same type of thing happens if you want to find a power series equal to $\int f(x)\,dx$. But what can we say about the intervals of convergence for the series we get?

7. Find the intervals of convergence for the following series:

$$\text{(a)} \quad \sum_{k=0}^{\infty} kx^{k-1} \qquad\qquad \text{(b)} \quad \sum_{k=0}^{\infty} x^k \qquad\qquad \text{(c)} \quad \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

8. Find the intervals of convergence for the following series:

$$\text{(a)} \quad \sum_{k=1}^{\infty} (-1)^k x^{k-1} \qquad\qquad \text{(b)} \quad \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k} \qquad\qquad \text{(c)} \quad \sum_{k=1}^{\infty} \frac{(-1)^k x^{k+1}}{k(k+1)}$$

9. What can you say about the *radius* of convergence when you differentiate or integrate a power series (see the top of page 668 of your text). What can you say about how the *interval* of convergence may change when you take the derivative or integral of a power series?

10. (Bonus) Can you explain why the radius of convergence remains unchanged when you differentiate or integrate a power series? (Hint: use the Ratio Test somehow.)

11. Seeing as we've just passed Pi Day, can you use your answer to Problem 5 to come up with a series that converges to π?

Preparation for next time

After the break, we will spend a week going over Sections 8.7 and 8.8 on Taylor series. For Monday, March 30, please read Section 8.7 from the beginning through Example 7.2. Work Exercise 1 from the section, showing all your work, and write down a reading question.

Today's random quote from the Mathematical Quotations Server at Furman University

"...in a few years, all great physical constants will have been approximately estimated, and ... the only occupation which will be left to men of science will be to carry these measurements to another place of decimals."
—James Clerk Maxwell (1813-1879)
Scientific Papers 2, 244, October 1871.