Merit Worksheet #23, 3/30/09

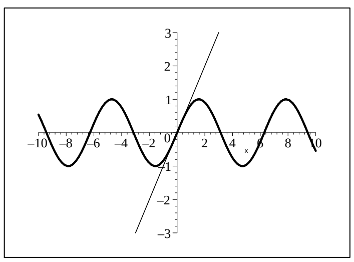
Polynomials that approach functions

1. (a) Use what you learned in Calc I (and/or something from Section 3.1 of your text) to find the equation of a line approximating $y = \sqrt{x+1}$ around the point $(0, 1)$. Carefully sketch both graphs on the same set of axes.

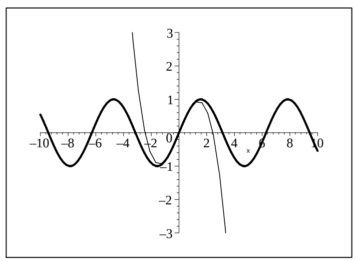
   (b) Find the equation of a *parabola* that approximates $y = \sqrt{x+1}$ about the point $(0, 1)$. In other words, find out what a, b, and c should be so that if $f(x) = ax^2 + bx + c$, then $f(x)$ and $\sqrt{x+1}$ have the same value **and** first derivative at $x = 0$. Carefully sketch both graphs on the same set of axes.

   (c) What could you do if you wanted to find the equation of a *cubic* function that approximates $y = \sqrt{x+1}$ about the point $x = 1$?

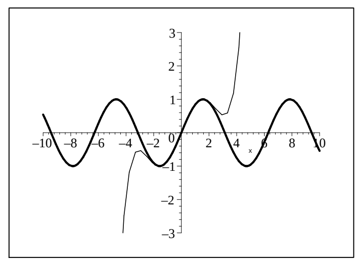
Remember how last time we saw how the graphs of the partial sums of $\sum x^k$ converged to the graphs of $1/(1-x)$? Well, here's some more of the same. The graphs below show the function $y = \sin x$ together with 7 different polynomials. Use these to answer the questions that follow.



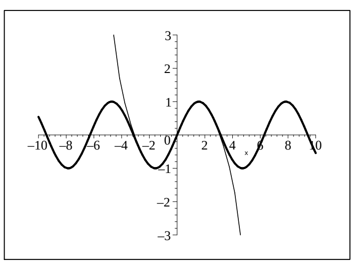
$y = x$



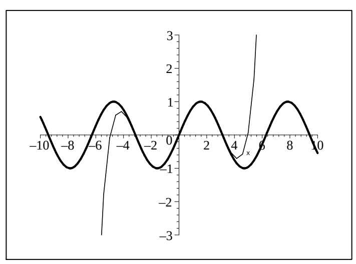
$y = x - \frac{1}{6}x^3$



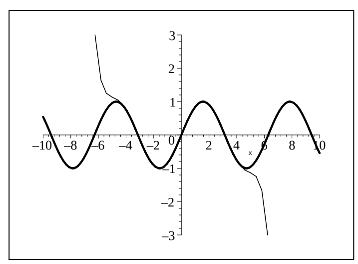
$y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$



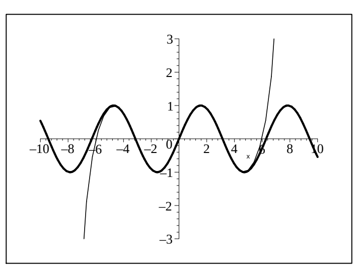
$y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$



$y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9$



$y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11}$



$$y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11} + \frac{1}{6227020800}x^{13}$$

2. (a) As you move through the figures, what do you notice about *graphs* of the polynomials? What do you notice about the polynomials themselves?
   (b) In each picture, note approximately on which interval you can't tell the difference between the polynomial and $y = \sin x$.

3. (a) Show that $y = x$ has the same value as $\sin x$ at $x = 0$, as well as the same first derivative value at $x = 0$, but it has a different value for its second derivative there.
   (b) For how many derivatives do $x - \frac{1}{6}x^3$ and $y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ agree with $\sin x$ at $x = 0$?
   (c) Can you guess a pattern, then, for the relationship between the derivatives at $x = 0$ of $\sin x$ and the polynomials whose graphs are shown?

1. Let's step away from the pictures now for a quick refresher. In your own words, what's a power series? What's the interval of convergence of a power series?

4. So, from your reading, what's a Taylor series? What's a Maclaurin series? What's the Taylor polynomial of degree n at $x = c$?

5. Where did the polynomials used in the graphs above come from?

6. Find the Taylor series about the given point and the interval of convergence of the series, for the following functions:

   (a) $f(x) = \dfrac{1}{(1+x)^2}, \quad x = 0;$ (b) $f(x) = \cos 2x, \quad x = \pi;$ (c) $f(x) = \ln x, \quad x = e.$

8. Find the Taylor series for the function $y = x^4 + 4x^3 + 3x^2 - 2x + 4$
   (a) about $x = 0$;
   (b) about $x = -1$.

9. Say you want to approximate $\sqrt{1.1}$.
   (a) Find the Taylor polynomial of degree 3 about $x = 0$ for the function $y = \sqrt{x+1}$. (Does your answer make sense in light of Problem 1?)
   (b) Use a calculator (someone in your group's, or find someone in the class who'd be willing to share) and your answer to part (a) to approximate $\sqrt{1.1}$. How close are you to the real square root?
   (c) Why was it good to find the Taylor polynomial about $x = 0$, rather than about, say, $x = -1$? (Can you think of more than one reason?)

10. (a) Compute the Taylor series for $\ln x$ around $x = 1$.
    (b) What is the interval of convergence for this series?
    (c) Why does your answer to part (b) make sense in terms of the graph of the function (what happens at $x = 0$)?
    (d) Using what you know about alternating series, give two numbers between which $\ln(1.5)$ must lie.

11. As a group, sum up the key ideas of this worksheet in a couple of sentences. Is there anything you're unclear on?[1] What would you guess that you'll talk about on Wednesday?

[1] Office hours are canceled until Thursday of this week, but I'm available by email!

Preparation for next time

We saw today that Taylor polynomials can approximate a function. *Be sure to try to get through this entire worksheet!* On Wednesday you'll discuss the *remainder term*, which will give us an idea of how far off such an approximation can be. In preparation for class, please read Theorem 7.1 and the paragraph that follows it on page 675, and then skip over to Examples 7.6 and 7.7 and read those. Prepare Problem 27 to turn in, along with a reading question.

Quote of the day

"On two occasions I have been asked [by members of Parliament], 'Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?' I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question."
— Charles Babbage (1792-1871), an English mathematician and mechanical engineer who (see wikipedia) "originated the idea of a programmable computer" and "is credited with inventing the first mechanical computer that eventually led to more complex designs."