Merit Worksheet #24, 4/1/09

Taylor series, polynomials, and remainder terms

1. Find the Taylor series about the given point and the interval of convergence of the series, for the following functions:

   (a) $f(x) = \ln x, \quad x = 1$ (b) $f(x) = e^{-x}, \quad x = 0,$ (c) $f(x) = \frac{1}{x}, \quad x = -1.$

2. (a) Find the 5th degree Taylor polynomial $P_5(x)$ for $f(x) = x^3 - 3x^2 + 5x - 7$ about $x = 1$.
   (b) What is the Taylor series for $f(x) = x^3 - 3x^2 + 5x - 7$ about $x = 1$?
   (c) How can you verify that your answer to part (b) really does equal $f(x)$?

3. (a) Find the 4th degree Taylor polynomial $P_4(x)$ for the function $f(x) = \sqrt{x}$ about $x = 1$.
   (b) Use a calculator (someone in your group's, or find someone in the class who'd be willing to share) and your answer to part (a) to approximate $\sqrt{1.1}$. How close are you to the real square root?
   (c) Why was it good to find the Taylor polynomial about $x = 1$, rather than about $x = 0$? (Can you think of more than one reason?)

4. (a) What is the remainder term that goes with the Taylor polynomial you found in the previous problem? (See Theorem 7.1 on page 675 of your text.)
   (b) Use your answer to part (a) to decide how big the error between $P_4(1.1)$ and $\sqrt{1.1}$ could, theoretically, be.
   (c) How does your answer to part (b) compare to how far off $\sqrt{1.1}$ actually is from $P_4(1.1)$?

5. (a) Compute the Taylor series for $\ln x$ around $x = 1$.
   (b) What is the interval of convergence for this series?
   (c) Why does your answer to part (b) make sense in terms of the graph of the function (what happens at $x = 0$)?
   (d) Using what you know about alternating series, give two numbers between which $\ln(1.5)$ must lie.

6. (a) What is the Maclaurin series for $f(x) = e^x$?
   (b) What is the remainder term $R_n(x)$ that goes with the Taylor polynomial $P_n(x)$ of degree n for $f(x) = e^x$ about $x = 0$?
   (c) Show that $R_n(x)$ approaches 0 as n approaches infinity.
   (d) What does this mean about how the Maclaurin series and the original function are related?

7. The *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

   is important in many applications. For example, it is used in calculating the probability that measurements fall near the center of a bell curve. Compute and graph the fourth-order Taylor polynomial for $\operatorname{erf}(x)$ about $c = 0$.

8. (Bonus) An *even* function is one in which $f(x) = f(-x)$ for all x. These are exactly the functions which are symmetric about the y-axis. For example, $\cos x$ is an even function.
   (a) Show that $f^{(k)}(x) = -f^{(k)}(-x)$ whenever k is odd.
   (b) Use part (a) to show that the Maclaurin series for $f(x)$ only needs even exponents. (This explains again why $\cos x$ has a power series with only even exponents.)
   (c) Can you do something similar to parts (a) and (b) to show that the Maclaurin series for an odd function only needs odd exponents? (An *odd* function is one for which $f(-x) = -f(x)$ for all x.)

Preparation for next time

Friday we will conclude our discussion of Section 8.7 in the text by seeing some shortcuts for finding Taylor series. In preparation, please read Example 7.8 on pages 681 and 682 of your text. Prepare Exercise 35 and a reading question for turning in.

The quiz on Friday will cover Section 8.7. I will have the usual office hours tomorrow from 4 to 5 PM. I've missed you guys...I hope to see you there!

Quote of the day

"I admit that mathematical science is a good thing. But excessive devotion to it is a bad thing."
—Aldous Huxley, 1934.