Merit Worksheet #25, 4/3/09

Remainder terms

1. Find the third-degree Taylor polynomial $P_3(x)$ and the remainder term $R_3(x)$ for each of the following functions $f(x)$ about the point indicated.

$$\text{(a)} \quad f(x) = e^{2x}, \quad c = 0, \qquad\qquad \text{(b)} \quad f(x) = 1/x, \quad c = 1.$$

2. Find the third-degree Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x}$ expanded about $x = 4$. Also give the remainder term $R_3(x)$. How big could the error be if we use $P_3(x)$ to approximate $\sqrt{1/3}$?

3. (a) What is the Maclaurin series for $f(x) = e^x$?
   (b) What is the remainder term $R_n(x)$ that goes with the Taylor polynomial $P_n(x)$ of degree n for $f(x) = e^x$ about $x = 0$?
   (c) Show that $R_n(x)$ approaches 0 as n approaches infinity.
   (d) What does this mean about how the Maclaurin series and the original function are related?

4. In many applications where a number x is very small, the term "$\sin x$" is replaced by just an "x" in order to simplify calculations. For example, in the section "Physical pendulums" of the article "Pendulum (mathematics)" at Wikipedia, the equation

$$\alpha = -\frac{mgL\sin(\theta)}{I} \qquad \text{is modified by writing} \qquad \alpha \approx -\frac{mgL\theta}{I},$$

   and from there the derivation goes on.

   (a) Explain why the Maclaurin series for $\sin x$ supports such a practice.
   (b) The function x is actually equal to $P_2(x)$, the second-degree Taylor polynomial for $\sin x$. Why?
   (c) What is the remainder term $R_2(x)$?
   (d) How big could the error be if we use $P_2(x)$ to approximate $\sin 1^\circ = \sin(\pi/180)$?

Series representation manipulations

5. What are the Taylor series expansions and intervals of convergence for $1/(1-x)$, e^x, $\sin x$, and $\cos x$ about $x = 0$, and the expansion for $\ln x$ about $x = 1$? **These are the series and intervals that you will need to know for the next quizzes, exam, and final.**

6. Find the power series for the following functions:

$$\text{(a)} \ \frac{1}{1+3x} \qquad \text{(b)} \ e^{-x^2} \qquad \text{(c)} \ \frac{\sin x}{x} \qquad \text{(d)} \ \frac{1-\cos x}{x} \qquad \text{(e)} \ \sqrt{1-x^2}$$

   What is the interval of convergence in each power series?

7. (a) Quick! From what you learned in first-semester calculus, what value do the following limits have?

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \qquad\qquad \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta}$$

   (b) How do your answers in parts (c) and (d) of Problem 6 support this?

8. Find a power series for the function

$$\frac{1}{(1+3x)^2}$$

(Hint: Use your answer to Problem 6(a) and take a derivative.) Without using the ratio test, can you state what the radius of convergence is?

9. The function $f(x) = e^{-x^2}$ does not have an easily writable antiderivative. The best we can do is say that the function $F(x) = \int_0^x e^{-t^2}\,dt$ is an antiderivative. Find a power series representation for $F(x)$. What is the interval of convergence for the series?

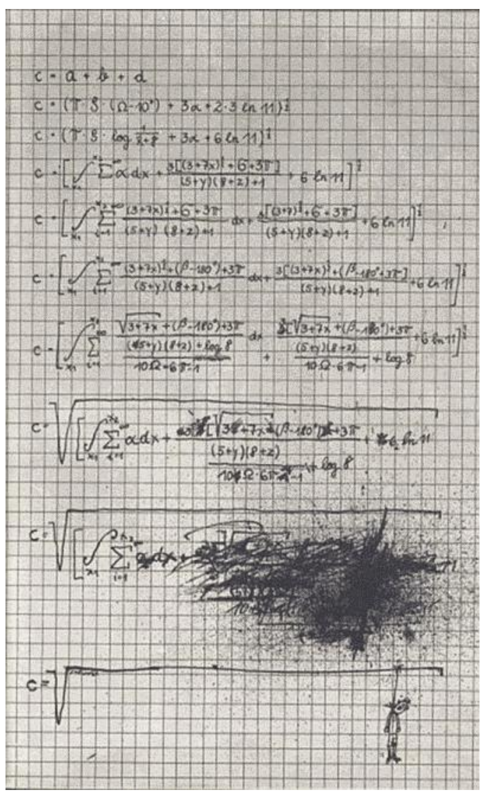
10. Using your answer to Problem 9 and a calculator, give an approximation to $\int_0^1 e^{-x^2}\,dx$ accurate to three decimal places.

Preparation for next time

Monday we will begin our discussion of Section 8.8. This section contains many applications of Taylor series. We've already talked about one—using Taylor series to approximate functions, like in Problems 2 and 4 above. In preparation for Monday's class, read about some other applications in Examples 8.3, 8.4, and 8.5 on pages 687–689 of your text. Prepare Exercises 7 and 15 and a reading question to turn in.

Joke of the day

Math got you down?



(http://www.mitadmissions.org/topics/misc/miscellaneous/everybody_loves_a_math_joke.shtml)