Merit Worksheet #26, 4/6/09

What you need to have memorized

Starting on the quiz this Friday, I'll expect you to have memorized the Maclaurin series for e^x, $\sin x$, and $\cos x$, and the Taylor series about $x = 1$ for $\ln x$. You should also know the interval of convergence for each series. This can all be found in the table on page 681 of your text.

Series representation manipulations

1. What are the Taylor series expansions and intervals of convergence for $1/(1-x)$, e^x, $\sin x$, and $\cos x$ about $x = 0$, and the expansion for $\ln x$ about $x = 1$? (Have EVERYONE in your group write up one of these on the board, and do it within the next 3 minutes!)

2. Using your answers to Problem 1, find the power series for the following functions:

   (a) $\dfrac{1}{1+3x}$ (b) e^{-x^2} (c) $\dfrac{\sin x}{x}$ (d) $\dfrac{1-\cos x}{x}$

   What is the interval of convergence in each power series? (Hopefully you've done these problem the quick way—what would be the long way to do these problems?)

3. (a) Quick! From what you learned in first-semester calculus, what value do the following limits have?

   $$\lim_{x\to 0} \frac{\sin x}{x} \qquad\qquad \lim_{x\to 0} \frac{1-\cos x}{x}$$

   (b) How do your answers in parts (c) and (d) of Problem 2 support this?

4. Use a known Taylor series to conjecture the limit

   $$\lim_{x\to 0} \frac{\sin(x^2) - x^2}{x^6}.$$

   Then use l'Hospital's rule to find the same limit. Which way did you prefer?

5. Find a power series for the function

   $$\frac{1}{(1+3x)^2}$$

   (Hint: Use your answer to Problem 2(a) and take a derivative.) Without using the ratio test, can you state what the radius of convergence is?

6. The function $f(x) = e^{-x^2}$ does not have an easily writable antiderivative. The best we can do is say that the function $F(x) = \int_0^x e^{-t^2}\,dt$ is an antiderivative.[1] Find a power series representation for $F(x)$. What is the interval of convergence for the series?

7. Using your answer to Problem 6 and a calculator, give an approximation to $\int_0^1 e^{-x^2}\,dx$ accurate to three decimal places.

8. The function $f(x) = \dfrac{\sin x}{x}$ does not have an antiderivative that can be written down in terms of nice functions. What's more, it's discontinuous at $x = 0$, so finding

   $$\int_0^1 \frac{\sin x}{x}\,dx$$

   isn't a trivial matter. In the next few parts, we'll do it anyway (kind of).

[1] Do you remember why? Hint: Look up the Fundamental Theorem of Calculus, Part II on page 387 of your text.

(a) From the Maclaurin series for $\sin x$, what would you guess that the limit of $(\sin x)/x$ should be as x approaches 0?

(b) Replace $(\sin x)/x$ by its 8th degree Taylor polynomial and integrate to find an approximation for the integral.

(c) The actual value of this integral, according to my computer, is really close to 0.9460830704. If someone in your group has a calculator, calculate numerically what you got for (b), and compare it to this value. How close were you?

9. Use power series, rather than l'Hospital's rule to evaluate the given limits:

$$\text{(a)} \quad \lim_{x \to 0} \frac{1 + x - e^x}{x^2} \qquad\qquad \text{(b)} \quad \lim_{x \to 0} \frac{x - \sin x}{x^3 \cos x}.$$

10. Say we wanted to approximate $\displaystyle\int_0^1 \arctan x \, dx$.

(a) Using nothing but the power series for $1/(1-x)$ and your bare hands, find a power series for $\tan^{-1}(x)$. Then use this power series to approximate the integral.

(b) Do something smarter than part (a) by using a method from Chapter 6 to evaluate the integral exactly.

11. The alternating harmonic series is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}.$$

We've seen earlier that it converges, by the alternating series test. Write out the Taylor series for $\ln x$ about $x = 1$, and see what it suggests about what this series converges to.

12. Use a known Taylor polynomial with 4 nonzero terms to approximate $\displaystyle\int_0^1 e^{\sqrt{x}} \, dx$.

13. What is the interval of convergence for the Taylor series for $\ln x$ about $x = 1$? Use this series, and the properties of the function $\ln x$, to approximate $\ln 3$.

Preparation for next time

Because of time concerns, we will not be covering the binomial series, which is found towards the end of Section 8.8; there's not much to it—it's just another power series, though it's very useful in practice, and I'd encourage you to read up on it (pages 691 to 692 of your text) for your own personal enrichment. On Wednesday well begin Chapter 9 by covering (all of) Section 9.1 on parametrically defined curves. Please read the first two examples in detail, and skim the rest of the section. Prepare Exercises 1 and 3 to turn in along with a reading question.

A very pad pun and a quote

Q: Why do truncated Maclaurin series fit the original function so well?
A: Because they are Taylor made.

"Our federal income tax law defines the tax y to be paid in terms of the income x; it does so in a clumsy enough way by pasting several linear functions together, each valid in another interval or bracket of income. An archeologist who, five thousand years from now, shall unearth some of our income tax returns together with relics of engineering works and mathematical books, will probably date them a couple of centuries earlier, certainly before Galileo and Vieta." —Hermann Weyl(1885 - 1955)