Merit Worksheet #28, 4/4/08

Parametric equations

1. Suppose two particles travel according to the parametrically defined curves given below, where $t \geq 0$:

$$\begin{cases} x &= \frac{16}{3} - \frac{8}{3}t, \\ y &= 4t - 5, \end{cases} \qquad \begin{cases} x &= 2\sin\frac{1}{2}\pi t, \\ y &= -3\cos\frac{1}{2}\pi t. \end{cases}$$

   (a) At what points, if any, do the paths intersect?

   (b) At what points, if any, do the particles collide?

2. (Bonus—come back to this) Find the places, if any, where the curve

$$x = \sin(2\pi t), \qquad y = 2t - t^2, \qquad 0 \leq t \leq 4,$$

   intersects itself.

3. Consider the curve $x = \cos t$, $y = \sin^2 t$, $-\pi \leq t \leq \pi$.

   a) Eliminate the parameter in the curve.

   b) Sketch the curve.

   c) Describe the motion of the point $(x(t), y(t))$ as t varies in the interval $-\pi \leq t \leq \pi$. (Be careful!)

4. Find an equation for the tangent to the curve

$$x = t^3, \qquad y = 1 - t, \qquad -\infty < t < \infty.$$

   at the point $(8, -1)$.

5. Find the points on the curve

$$x = 3 - 4\sin t, \qquad y = 4 + 3\cos t, \qquad -\infty < t < \infty$$

   where the tangent line is either horizontal or vertical (and state which are which).

6. The position of an object is given by

$$x = 2\cos t, \qquad y = 2\sin t, \qquad 0 \leq t \leq 2\pi.$$

   Find the object's velocity (horizontal and vertical components) **and** speed at $t = 0$ and at $t = \pi/2$.

7. Consider the curve $x = e^t$, $y = e^{-t}$.

   (a) Find a formula for dy/dx (your answer will involve t).

   (b) Find d^2y/dx^2 (again, the answer will involve t).

   (c) Eliminate the parameter and write the curve's equation in terms of x and y.

   (d) Find dy/dx and d^2y/dx^2 just using your answer to part (c). How do these derivatives compare to the ones you found in parts (a) and (b)?

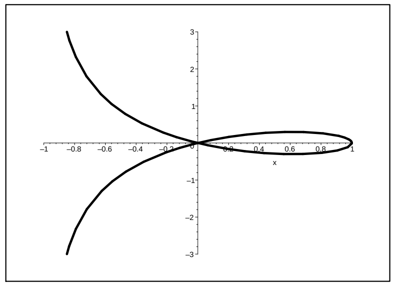
8. Suppose a particle's path is given by

$$x = 3t - 7, \qquad y = t^2 - 2t + 4, -\infty < t < \infty.$$

   Find the speed of the particle as a function of t. Is there ever a *fastest* speed the particle attains? Is there ever a *slowest* speed the particle attains? If either of these happens, what's the speed, and when does it happen?

9. Shown below is the graph of the parametric equations

$$x = \frac{1-t^2}{1+t^2}, \qquad y = \frac{t(1-t^2)}{1+t^2}, \qquad -\infty < t < \infty.$$



This curve is called a *strophoid*. It has a nifty geometric definition which we won't go into here, but which you can find online or ask me about.

   (a) Find the slope of the tangent lines to the curve at the origin.
   (b) Find the points at which the tangent line is horizontal.
   (c) Find dy/dx, and describe how the curve behaves as t approaches $\pm\infty$.

10. By now we've seen that circles and ellipses can be parametrized using sine and cosine functions. However, there's also a parametrization of the unit circle involving simpler functions that apparently was known to the ancients. It's the following:

$$x = \frac{1-t^2}{1+t^2}, \qquad y = \frac{2t}{1+t^2}, \qquad -\infty < t < \infty.$$

   (a) Find a formula for dy/dx in terms of t.
   (b) Find a formula in terms of t for the speed at which the curve is traced out. If you were to watch a movie of the circle being traced out, how would it look?
   (c) (Bonus) Suppose you're given the unit circle and a line with slope t that passes through the point $(-1, 0)$. Show that the line intersect the unit circle at the point (x, y) given by the parametric equations above (this is how we derive the equations).

Preparation for next time

On Monday we'll finish Section 9.2 by learning about the area enclosed by a parametric curve. Please read from Theorem 2.2 on page 730 through the end of the section, and then prepare Exercise 21 and a reading question to hand in.

Quotes of the day

" 'Every minute dies a man, Every minute one is born;' I need hardly point out to you that this calculation would tend to keep the sum total of the world's population in a state of perpetual equipoise, whereas it is a well-known fact that the said sum total is constantly on the increase. I would therefore take the liberty of suggesting that in the next edition of your excellent poem the erroneous calculation to which I refer should be corrected as follows: 'Every moment dies a man, And one and a sixteenth is born.' I may add that the exact figures are 1.067, but something must, of course, be conceded to the laws of metre." —Charles Babbage, letter to Alfred, Lord Tennyson, about a couplet in his "The Vision of Sin."