Merit Worksheet #29, 4/13/09

Parametric equations

1. Suppose we have a parametric curve traced out by a particle whose location is given by $x(t)$ and $y(t)$. Give formulas for, or explain how you would find, each of the following quantities:

   (a) The horizontal and vertical components of the particle's velocity;

   (b) The particle's speed;

   (c) The slope of the curve's tangent line, i.e., dy/dx;

   (d) The second derivative at a point on the curve;

   Refer to section 9.2 in your text if necessary.

2. Remember back in Calc I when we wanted to find the area under a curve? We split the region we wanted up into very thin rectangles and added them up (in a Riemann sum) to get the area. In this problem, we'll talk about the area under a parametric curve.

   (a) Sketch the graph of a random positive function $f(x)$. In your drawing, split up the area under the curve into thin rectangles, as if you were about to do a Riemann sum.

   (b) Based on your drawing, *why* is the formula

   $$\text{Area under } f(x) = \int_a^b f(x)\,dx$$

   correct? How do a, b, $f(x)$, and dx all show up on the graph?

   (c) How do you calculate y and dx if you're given a curve that's defined parametrically?

   (d) Based on your answers to parts (a), (b), and (c), what's a formula for the area under a parametric curve traced from left to right as t goes between two numbers c and d?

   (e) The area formula given on page 730 looks a little different, and it's talking about something different. Explain the difference between the area formula there and what you found in part (d). **On homework, quizzes and tests, I will expect you to know the formula you figured out in (d). Focus on memorizing this rather than the formulas on page 730; using an incorrect formula from page 730 will penalize you.**

3. Consider the curve traced out by $x = \cos^3 t$, $y = \sin^3 t$.*

   (a) Find the area under the curve between $t = 0$ and $t = \pi/2$.

   (b) Find a formula for dy/dx (your answer will involve t).

   (c) At what speed is the curve traced out when $t = \pi/4$?

   (d) At what slope is the curve being traced out when $t = \pi/4$?

   (e) What's the second derivative of the curve when $t = \pi/4$?

4. Suppose we're given the curve $x = 3t - t^3$, $y = -t^2/10 + t + 1$.

   (a) Find any points on the curve where the tangent line is horizontal or vertical.

   (b) Find the second derivative of the curve at the point $(0, 1)$.

5. Suppose $x = t^2 - 2t$, $y = t^3 - 12t$.

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*This curve, called the *astroid*, appears prominently in Project #1, which is now available on the projects webpage.

(a) Find the area under the curve between $t = -2$ and $t = 0$.

(b) Find the area above the curve between $t = 2$ and $t = 3$.

6. Taking the same curve as in Problem 5, come up with a detailed sketch of it. Make sure your sketch accurately shows each of the following:

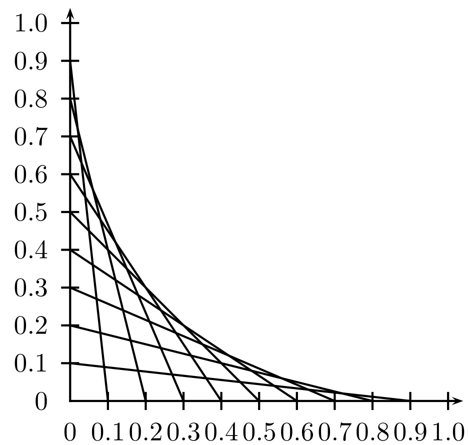
   (a) x- and y-intercepts of the curve;

   (b) horizontal and vertical tangents to the curve;

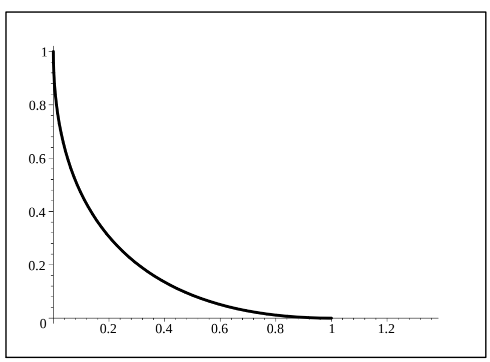
   (c) places where the curve is concave up and concave down;

   (d) How the curve behaves as t approaches $\pm\infty$.

7. Suppose that for every point A on the x-axis you calculated $B = 1 - A$ and found B on the y-axis, and then drew a straight line connecting A and B. If you drew enough of these straight lines, you might get something like the following:



See how the lines combine to produce sort of a curve? That curve is called the *envelope* of the lines, and it's graphed below:



This curve has parametric equations $x = t^2$, $y = (1-t)^2$, $0 \le t \le 1$.[†]

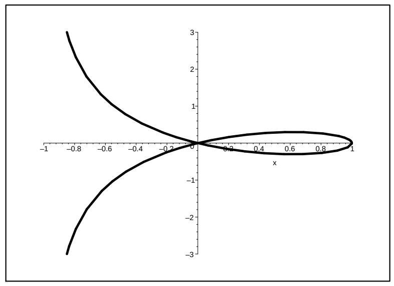
   (a) Find the area of the region enclosed by this curve and the coordinate axes.

[†]I won't include the details of how you can find out what this curve's equation is (or make you do it yourselves), but it's possible to find the equation in rectangular coordinates using only first-semester calculus. You might try it! One important thing to note: Though it looks similar, this curve is NOT the same as the *astroid* mentioned in Problem 3 and Project #1.

(b) Find the length of this curve. (Hint: how are speed and distance related? What can you say about the speed at which the curve is traced out?)

8. Shown below is the graph of the parametric equations

$$x = \frac{1-t^2}{1+t^2}, \qquad y = \frac{t(1-t^2)}{1+t^2}, \qquad -\infty < t < \infty.$$



This curve is called a *strophoid*. It has a nifty geometric definition which we won't go into here, but which you can find online or ask me about.

(a) Find the slope of the tangent lines to the curve at the origin.

(b) Find the points at which the tangent line is horizontal.

(c) Find dy/dx, and describe how the curve behaves as t approaches $\pm\infty$.

9. Suppose we've got a circle of radius R. Find a parametrization for this circle, and use it to prove that the area enclosed is πR^2.

10. By now we've seen that circles and ellipses can be parametrized using sine and cosine functions. However, there's also a parametrization of the unit circle involving simpler functions that apparently was known to the ancients. It's the following:

$$x = \frac{1-t^2}{1+t^2}, \qquad y = \frac{2t}{1+t^2}, \qquad -\infty < t < \infty.$$

(a) Find a formula for dy/dx in terms of t.

(b) Find a formula in terms of t for the speed at which the curve is traced out. If you were to watch a movie of the circle being traced out, how would it look?

(c) (Bonus) Suppose you're given the unit circle and a line with slope t that passes through the point $(-1, 0)$. Show that the line intersect the unit circle at the point (x, y) given by the parametric equations above (this is how we derive the equations).

Preparation for next time

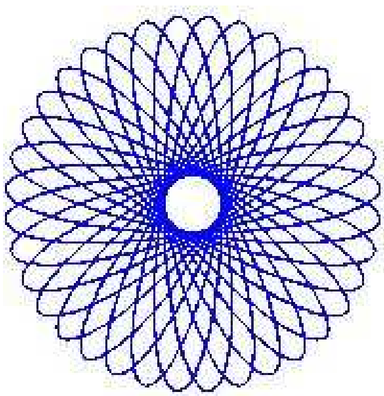
On Wednesday we'll cover Section 9.3, discussing how to find the lengths of parametric curves and the surface area of solids of revolution. Please read Theorem 3.1 on page 735 and Examples 3.1, 3.2; then read from halfway down page 738 ("Much as we did in section 5.4...") through the boxed formula at the top of page 739, followed by Example 3.6. Turn in a reading question and the following problems:

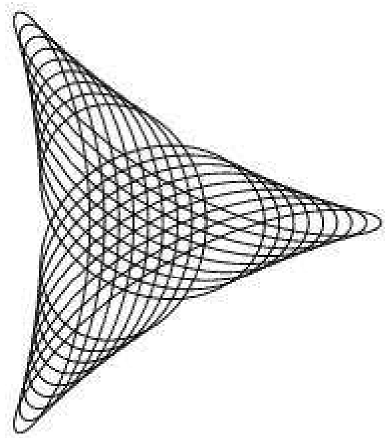
- **Using the parametric formula for arc length, find the length of the curve $x = t^2$, $y = t^3$ between $t = 0$ and $t = 1$.**
- **Find the surface area when the curve $x = r\cos t$, $y = r\sin t$, $0 \le t \le \pi$ is revolved around the x-axis.**

“Math is cool” moment of the day:

So, you’re asking yourself, when will I ever use parametric equations? Well, let me ask you a question: have you ever seen this toy?

A Spirograph, in case you’re not familiar with it, is a toy put out by Hasbro in which you stick your pen through a hole in a plastic gear (which is usually shaped like a circle, though not always). You roll your circle gear around a frame (which is also often a circle), and as you do so your pen traces out neat designs on paper, such as the ones below:





These curves have been studied before; they even have technical names: the curves you get by rolling your circular gear around the inside of a circular frame are called *hypotrochoids*, and those curves you get by rolling your gear around the outside of the frame are called *epitrochoids*. So how could someone ever come up with an equation for a curve as complicated as one of these? By using parametric equations, of course. You can do it yourself in Problems 41 and 42 from Section 8.2.

Even if you’re not interested in messing with the equations of a Spirograph curve, I heartily recommend learning more about them. Start by glancing over the parts of the text mentioned above, and then check out the articles for “spirograph” at Wikipedia and www.mathworld.com. There are also lots of online applets that mimic a Spirograph’s action and trace out curves of your choosing; try finding one of those. And, of course, the *truly* enthusiastic can go to www.hasbro.com to purchase their very own Deluxe Spirograph design toy (ages 5 and up, approx. retail value \$7.99). If you get one, let me know if you like it. Mine came from a garage sale, and it’s getting a little old, so I’m wondering whether I should get a new one.

So anyway, there’s one application of parametric equations—with them you can understand all you want to about those curves you drew as a kid. You can find their slopes, their lengths, the areas of the region(s) they enclose, the volumes of the solids you’d get by rotating them about the coordinate axes, ...

Images from `www.samstoybox.com/toys/Spirograph.html`, `www.thepcmanwebsite.com/media/spirograph/spirograph.shtml`, and `http://mathworld.wolfram.com/Spirograph.html`, respectively.