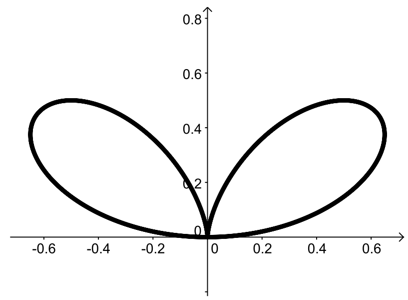
Merit Worksheet #34, 4/27/09

A bit more of calculus and polar curves

1. Find the area inside the graph of $r = 2\sin\theta$ but outside the graph of $r = 1$.

2. Shown below is the graph of $r = \sin 2\theta \cos\theta$.



In what follows, it may help to remember that $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

(a) If θ starts at 0, where should we *stop* θ so that the curve is traced out exactly once between the starting and stopping points?

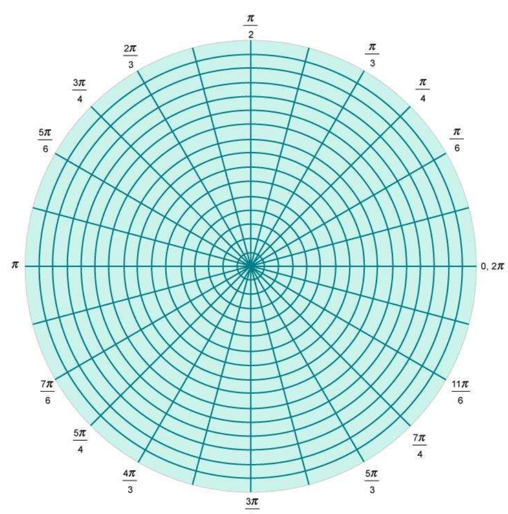
(b) Label the graph with arrows at a few points to make it clear in which direction the graph is traced out.

(c) What is the slope of the tangent line to the graph when $\theta = \pi/2$? When $\theta = 2\pi/3$?

(d) Which point or points on the curve are the farthest away from the origin? (For each point, give both polar coordinates, or both cartesian coordinates.)

(e) How much area is enclosed by one loop of the graph?

3. Sketch the graph of $r = 1 + \cos\theta$. This graph is called a *cardioid*.



(a) On your graph, indicate in which direction the graph is traced out.

(b) Find all points where the tangent line to the curve is horizontal or vertical. (For each point, give both polar coordinates, or both cartesian coordinates.)

(c) Find the area enclosed by this graph.

4. Find the area inside the graph of $r = 2\sin\theta$ but outside the graph of $r = 1$.

5. When you first learned in Calc I that the area under the curve $y = f(x)$ between $x = a$ and $x = b$ is $\int_a^b f(x)\,dx$, the formula was explained to you by drawing very thin rectangles and adding their areas together. Informally speaking, each rectangle had height $f(x)$ and width dx, so the formula $\int f(x)\,dx$ made sense. How in the world do we come up with the formula for polar area? How is the area inside a polar curve being sliced up? Be sure to draw a diagram on the board and give a *good* explanation.

6. The curve whose equation in rectangular coordinates is

$$y^2 = x^2\left(\frac{a-x}{a+x}\right), \qquad a > 0,$$

is called a strophoid.

(a) Based on its equation in rectangular coordinates, how would you guess that this curve behaves near $x = -a$?

(b) Show that the polar equation of this curve has the form $r = a\cos 2\theta\sec\theta$.

(c) Come up with a rough sketch of this curve.

(d) This curve has a loop. Find the area inside the loop when $a = 2$.

7. The curve whose equation in rectangular coordinates is

$$(x^2+y^2)^2 = ax^2y, \qquad a > 0,$$

is called a bifolium.

(a) Show that the polar equation of this curve has the form $r = a\sin\theta\cos 2\theta$.

(b) Come up with a rough sketch of this curve.

(c) Find the area inside one of the curves loops when $a = 2$.

8. Come up with a formula for the length of the polar curve $r = f(\theta)$ between $\theta = a$ and $\theta = b$.

9. Come up with cartesian-coordinate forms of the polar equation

$$r = \frac{ae}{e\cos\theta + 1},$$

where a and e are constants. What kind of curve do you get when

(a) e is between 0 and 1?

(b) $e = 1$?

(c) $e > 1$?

How are these curves related?

Preparation for next time

Next time we will cover Sections 9.6 and 9.7 on the conic sections. It's interesting stuff; all of Section 9.6 is worth reading, if you have the time, but at the least you should prepare Problem 1 of Section 9.6 and a reading question to be turned in. Next time will be the last of the new stuff. After that, we'll spend 3 days reviewing for the final.

Mathematical sidenote of the day

You may have wondered how I've assigned groups this semester (when I didn't let you do it yourselves). As I've mentioned before, I use a computer to make up the groups in a somewhat random manner. How random? *Truly* random.

You might ask how it's possible to get truly random numbers with a computer. After all, someone had to program the computer to generate the numbers, so how can they really be random? How can you design a purely random outcome? Even flipping a coin doesn't work perfectly—it's been shown recently that when you flip a coin, the coin is slightly more likely to land oriented the way it was before you flipped it. So what's a randomness-hungry person to do?

Enter `www.random.org`, an online random number generating service. The way `random.org` generates its random numbers is to let nature do it; according to the website,

"RANDOM.ORG offers true random numbers to anyone on the Internet. The randomness comes from atmospheric noise, which for many purposes is better than the pseudo-random number algorithms typically used in computer programs ... [True random number generators] extract randomness from physical phenomena and introduce it into a computer. You can imagine this as a die connected to a computer, but typically people use a physical phenomenon that is easier to connect to a computer than a die is... [A] suitable physical phenomenon is atmospheric noise, which is quite easy to pick up with a normal radio. This is the approach used by RANDOM.ORG... Regardless of which physical phenomenon is used, the process of generating true random numbers involves identifying little, unpredictable changes in the data. For example, ... RANDOM.ORG uses little variations in the amplitude of atmospheric noise."

You've seen and heard the randomness of atmospheric noise—it's the cause of the annoying static you hear between radio stations or see when your TV's turned to a channel it doesn't get. An article in *Science News* says the following:

"Random.org uses a radio to pull random numbers out of the atmospheric noise generated by weather systems.

" 'When we built this in 1997, we bought the cheapest radio we could find,' says Mads Haahr, of Trinity College, Dublin, who runs Random.org. 'The guy in the shop thought we were crazy because we made him take it out of the box and put in batteries so we could listen to the noise between stations.'

"Haahr's Web site (http://www.random.org/) can generate up to 3,000 random numbers per second. Over the last 6 years, it has dished out more than 61 billion random numbers for free to an eclectic array of users. These include archaeologists choosing which quadrants of a large area to survey; a choreographer selecting the order, timing, and placement of dance steps; online card-playing sites shuffling their virtual decks; the U.S. Environmental Protection Agency determining which companies to include in a random audit of hazardous-material use; and a locksmith deciding how deeply to cut the notches on keys."

And, of course, your Math 231 instructor, in determining how to arrange you into groups.

If you'd like to read more about random.org or other random number generators, please visit one of the many articles linked to at `http://www.random.org/media/`.

Review problems for the final — Sections 8.8 and 9.1

A. Use at least three terms of an appropriate Taylor series to approximate $\sqrt{65}$.

B. Without using l'Hospital's rule, conjecture the value of the following limit:

$$\lim_{x\to 0} \frac{x - x^3/6 - \sin x}{2x^5}.$$

C. Use a Taylor polynomial with at least five terms to approximate $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$.

D. Find the first five terms of the Maclaurin series for the following functions. (You may start from a known power series, if it would help.)

$$\text{(a)}\quad \frac{1}{1+x^2} \qquad \text{(b)}\quad (1-x)^{1/3} \qquad \text{(c)}\quad x\cos(2x)$$

E. Using one of your answers to Question D above, find the first five terms of the Maclaurin series for $\tan^{-1} x$.

F. (a) Suppose a curve is given parametrically by $x(t) = 2t - 1$, $y(t) = 4t^2 - 5$. Eliminate the parameter to find an equation for the curve only in terms of x and y.

(b) Do the same for the curve $x = 3\cos 2t$, $y = 2 + \sin 2t$.

G. An object has its position given by

$$\begin{cases} x = 4 - |t|, \\ y = (t-1)^2, \end{cases}$$

Sketch the graph of the object's path on the interval $-5 \le t \le 5$. Describe in words what path the object takes, including the points at where it starts and stops, and the direction(s) in which it travels.