Merit Worksheet #37, 5/4/09

Review of Chapter 8

1. (a) What is a sequence?
   (b) What does it mean for a sequence to be increasing? Decreasing? Monotonic?
   (c) Determine whether or not the sequence $\{a_n\}_{n=1}^{\infty}$ converges, and find its limit if it does converge:

   (i) $a_n = \dfrac{1}{n^3}$ (ii) $a_n = \dfrac{n}{n+1}$ (iii) $a_n = 2 - \left(-\dfrac{1}{2}\right)^n$ (iv) $a_n = n \sin\left(\dfrac{1}{n}\right)$

   (v) $a_n = \dfrac{5n^3 - 1}{2n^3 + 1}$ (vi) $a_n = (-1)^n \dfrac{n+4}{n+1}$ (vii) $a_n = \dfrac{e^{2n} + 2}{e^n - 1}$ (viii) $a_n = \cos(\pi n)$

2. What is a series? What is a *term* of a series? What's a partial sum? What does it mean for a series to converge?

3. Which of the geometric series below converge? For those that converge, find the series' sum.

$$\sum_{k=0}^{\infty} (-1)^k \qquad \sum_{k=3}^{\infty} \frac{1}{2^k} \qquad \sum_{k=1}^{\infty} \frac{3^{2k-1}}{5^{k+1}} \qquad \sum_{k=1}^{\infty} \frac{1}{2} \qquad \sum_{k=2}^{\infty} \frac{5}{2^{k+1}}$$

4. Determine whether each of the following statements are true or false. If a statement is true, give an explanation justifying it; if it is not, give an example of a series that shows that it is false. (Note: One of these statements is called the kth-Term Test for Divergence. Which one is it?)

   (a) If $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ converges.

   (b) If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

   (c) If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

   (d) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\lim_{k\to\infty} a_k \neq 0$.

5. Use the integral test to decide whether the series converge or diverge:

$$\text{(a)} \sum_{k=1}^{\infty} \frac{4}{\sqrt[3]{k}} \qquad \text{(b)} \sum_{k=2}^{\infty} \frac{1}{k \ln k} \qquad \text{(c)} \sum_{k=4}^{\infty} \frac{2k}{k^2+1}$$

6. Decide whether the series below converge or diverge by using the comparison test:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{n^2+1} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

7. Decide whether the series below converge or diverge by using the limit comparison test:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{4n+1}{3n^3 - n^2 - 1}$$

8. Which of the following series converge, and why?

(a) $\sum_{k=1}^{\infty}(-1)^k \frac{2}{k^2}$ (b) $\sum_{k=1}^{\infty}(-1)^{k+1} \frac{k^2}{k+1}$ (c) $\sum_{k=1}^{\infty}(-1)^{k+1} \frac{k}{2^k}$ (d) $\sum_{k=1}^{\infty}(-1)^k \frac{4^k}{k!}$

9. One way to approximate π is to use the *Leibnitz formula*

$$\frac{\pi}{4} = \sum_{k=1}^{\infty}(-1)^{k+1} \frac{1}{2k-1}.$$

(a) Show that the series on the right converges.
(b) Suppose we took the 100th partial sum of this series. How far away from $\pi/4$ could that answer be?
(c) Suppose we wanted to compute $\pi/4$ to within $1/1000$ of its true value. Which partial sums would be good enough to achieve this?

10. (a) What does it mean for a series to converge absolutely? Conditionally?
(b) Which of the following series converge absolutely? Conditionally? Not at all?

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \qquad \sum_{k=1}^{\infty} \frac{1}{k} \qquad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

11. Use the ratio test to decide whether or not the following series converge:

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k k}{3^k}$ (b) $\sum_{k=1}^{\infty} \frac{10^n}{n!}$ (c) $\sum_{n=1}^{\infty} \frac{e^n}{n}$

12. Use the root test to determine whether or not the series below converge:

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k k}{3^k}$ (b) $\sum_{k=1}^{\infty} \left(\frac{3k-1}{4k+6}\right)^k$ (c) $\sum_{n=1}^{\infty} \frac{e^n}{n}$

13. Find the interval and radius of convergence:

(a) $\sum_{k=0}^{\infty} \frac{k}{6^k}(x-3)^k$ (b) $\sum_{k=1}^{\infty} \frac{1}{k^2} x^k$ (c) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ (d) $\sum_{k=1}^{\infty} \frac{\ln k}{k}(x+1)^k$

14. Find the first 5 terms of the Taylor series for the following functions about the given point:

(a) $f(x) = \frac{1}{(1+x)^2}$, $x = 0$; (b) $f(x) = \cos 2x$, $x = \pi$; (c) $f(x) = \ln x$, $x = e$.

15. (a) Find the 4th degree Taylor polynomial $P_4(x)$ for the function $f(x) = \sqrt{x}$ about $x = 1$.
(b) What is the remainder term $R_4(x)$ that goes with this Taylor polynomial?
(c) Use your answer to part (b) to decide how big the error between $P_4(1.1)$ and $\sqrt{1.1}$ could, theoretically, be.

16. Starting from known power series, find the first five terms of the power series for each of the following functions:

(a) $\frac{1}{1+3x}$ (b) e^{-x^2} (c) $\frac{\sin x}{x}$ (d) $\frac{1-\cos x}{x}$

What is the interval of convergence in each power series?

17. The function $f(x) = e^{-x^2}$ does not have an easily writable antiderivative. The best we can do is say that the function $F(x) = \int_0^x e^{-t^2}\, dt$ is an antiderivative. Find a power series representation for $F(x)$.

Quote of the day

"A thing is obvious mathematically after you see it."
— R. D. Carmichael (1879–1967), a mathematician who taught here at the U of I for many years.

Preparation for next time

On Wednesday we'll finish our review with the material from Chapter 9. Please look over everything you have (including the final review problems I've give you) for Chapter 9 (Parametric and polar curves). Make a list of whatever questions you have to be answered, and identify some specific problems that either leave you stumped, or that you'd just like to see the solution to. There will be nothing to turn in.