Merit Worksheet #3, 8/30/06

1. For what values of C does the polynomial

$$x^2 + 3x + C$$

have two real roots? One real root? No real roots?

2. In each case, give an example of a function as described or explain why no such function exists.

   a) A polynomial function of degree less than 2 whose graph lies entirely above the x-axis.
   b) A polynomial of positive degree whose graph lies entirely beneath the x-axis.
   c) A polynomial of positive degree whose graph lies entirely below the x-axis.
   d) A polynomial of odd degree whose graph does not intersect the x-axis.
   e) A polynomial whose graph lies entirely between the lines $y = -1$ and $y = 1$.
   f) A rational function that has both positive and negative values but is never zero.
   g) A nonconstant rational function that is never zero and has no vertical asymptote.

3. In each of the following five cases sketch the graph of a function of the type described.

   a) A quadratic polynomial with no real zeros.
   b) A cubic polynomial with exactly one real zero $x \neq 0$.
   c) A cubic polynomial with exactly two distinct real roots.
   d) A quartic polynomial with exactly two distinct real zeros.
   e) A quartic polynomial with exactly three distinct real zeros.

4. Look at the rational function

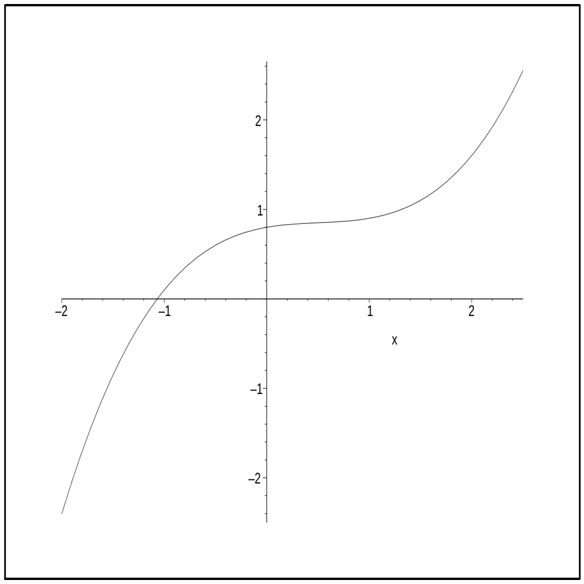
$$f(x) = \frac{x^6 + x^5 - 13x^4 - 13x^3 + 36x^2 + 36x}{2x^3 - 6x^2 + x + 3}.$$

What can you say about the number of zeros $f(x)$ might have? How many asymptotes could it possibly have?

5. At which place(s) does the rational function

$$\frac{2x^2 + 6x + 4}{x^2 - 3x - 4}$$

have an asymptote?



6. The graph of $f(x)$ is shown above. Suppose $g(x)$ is the inverse of $f(x)$, i.e. $f(g(x)) = x$. Sketch the graph of $g(x)$.

7. Sketch the graph of $f(x) = 3x + 2$. Let's say $g(x)$ is the inverse of $f(x)$. Sketch the graph of $g(x)$. What type of function is $g(x)$? Based on the graph, can you find a formula for $g(x)$?

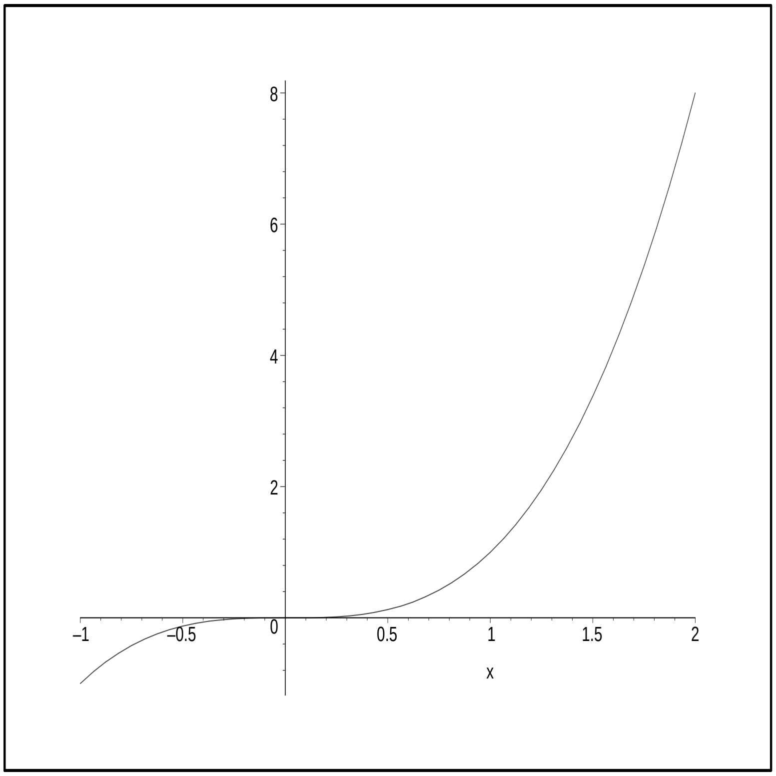
8. Give the value of x that makes each equation below correct. If an equation has no solution, explain why.

   a) $2^x = 8$
   b) $3^x = -1$
   c) $x = \log_2(-8)$
   d) $3 = \log_4(x)$
   e) $0 = \log_b(x)$

9. Suppose you drive from your apartment near campus down to the Savoy 16 and notice that your speedometer reads 45 mph at some instant of your trip. Since (a) you're not travelling 45 miles, (b) it doesn't take you an hour to get there, and (c) your speed is not constant during your drive, explain what exactly it means to be travelling at 45 miles per hour.

10. Consider the function $f(x) = x^3$, shown on the next page.

    a) Draw the line segments joining $(-1, -1)$ to $(2, 8)$, $(0, 0)$ to $(2, 8)$, and $(1, 1)$ to $(2, 8)$. What are the slopes of these line segments?
    b) Suppose h is some random constant. What is the slope of the line segment joining $(2, 8)$ and $(2 + h, (2 + h)^3)$?
    c) Are the line segments from part (a) pointing in the same direction as the function is at $(2, 8)$? If you wanted to get a line segment connecting $(2, 8)$ to some other point of the graph, so that the line segment was pointing in pretty much the same direction as the graph is at $(2, 8)$, where should you choose the other endpoint to be?



Tales of Death in Mathematics II: Another one bites the dust

As you know, there is a formula for solving any quadratic equation, and one way you can get that formula is by completing the square (remember how fun that was?). Similarly, it is possible through some tricks to solve any cubic equation or any quartic equation, though it's a lot more complex than solving a quadratic equation. Is it possible to find exact solutions for *any* polynomial equation? The answer is no, and this was proved by a young French mathematician named Évariste Galois (1811-1832).

Galois was a brilliant young mathematician whose life has been called a blend of genius and stupidity. As a young student he was so absorbed by mathematics that he didn't study for his other classes; as a result he failed key entrance exams for various schools. One story says he even threw an eraser at a teacher who was giving an entrance exam, because the teacher was asking stupid questions.

During the turbulent times in his own life and in France's political climate (Galois was a staunch member of the Republican artillery in the National Guard), Galois developed some amazing mathematics, including the tools necessary to prove that there is no formula to find the roots of any polynomial of degree 5 or higher. He tried to have his work published many times, but never succeeded while he was alive; one publisher is said to have lost it, and another to have died before getting a chance to publish it.

Finally, you'll notice that he didn't live very long. When he was 20 Galois entered into a duel, supposedly over a woman he was in love with. The night before the duel, convinced of his impending death, Galois stayed up writing letters to Republican friends and writing a summary of some of the mathematics he was currently working on. The next day he was shot and died some hours later. The moral? Don't stay up all night if you want quick reflexes the next day—even if it's for some math.

You can read more about Galois at http://en.wikipedia.org/wiki/%C3%89variste_Galois .

Selected answers

1. Two real roots for $C < 9/4$, one real root for $C = 9/4$, and no real roots for $C > 9/4$.
2. The following are only examples; there are many correct answers.
   a) Any positive constant function will work, like 3 or 7.
   b) $-x^2 - 1$, $-x^4 - 3$.
   c) See part (b)
   d) No such function exists.
   e) No such function exists.
   f) $1/x$
   g) $1/(x^2 + 1)$, x^2/x
4. $f(x)$ has no more than 6 zeros; it has no more than 3 vertical asymptotes.
5. At $x = 4$ (only!)
7. $g(x) = \frac{1}{3}x - \frac{2}{3}$
8. a) $x = 3$
   b) no solution
   c) no solution
   d) $4^3 = 64$
   e) no solution
9. It'd be great if you said something along the following lines: at that precise instant that your speedometer reads 45 mph, if you were to continue on at the same speed for an hour, you would travel 45 miles.
10. a) 3, 4, 7
    b)
    $$\frac{(2+h)^3 - 8}{(2+h) - 2} = \frac{(8 + 12h + 6h^2 + h^3) - 8}{h} = 12 + 6h + h^2.$$
    c) The line segments are not pointing in the same direction as the graph is. In order to get a line segment that points in a direction very close to the true one, you should put the other endpoint at a place on the graph ***very close*** to $(2, 8)$.