Merit Worksheet #8, 9/15/06

YOU DID IT! ONE TEST OUT OF THE WAY!

Grading will take place today (it's going on as we speak, and I'm missing out on that party to be with you. I hope you appreciate it), and you should get your tests back early next week.

Today's worksheet won't cover much new material, if any, but I hope you'll find these problems interesting.

1. What did you think of the test? Did you feel prepared? Write down which concepts, if any, you would have liked to have studied more about. **What will you do to better prepare for the next exam?** What suggestions will you give Mike so he can make the workshop more helpful for you?

2. Okay. Now for the math... You have got a parabolic milkshape glass, which conveniently happens to sit on the xy-plane and fits the graph of $y = x^2$. Instead of waiting until the end, you like to drop your cherry in the cup *before* it's filled with your milkshake. Now if you drop a small cherry in the cup, it will fall all the way to the bottom of the cup. If the cherry is too big, it won't fall all the way to the bottom, but will get stuck before it gets there. Your assignment: find the radius of the largest cherry that will fall all the way down. (If you'd like to see a picture of the setup, look at the diagram on page 100.)

3. We have a very easy way to tell if a quadratic equation has zero, one, or two solutions–we use the quadratic formula, and the *discriminant* $b^2 - 4ac$ in particular. Suppose you want to learn about a *cubic* function
$$f(x) = x^3 + px^2 + qx + r.$$
Answer the following questions.

   a) Some cubic functions, like $x^3 - 3x^2 + 1$, have two "bends" in their graphs (see page 27 for a picture of this graph); others, like x^3 and $x^3 + x$, are increasing the whole time and never bend downwards. Using the formula for $f(x)$ above, and the derivative $f'(x)$, how can you tell if $f(x)$ has two bends, or none?

   b) How many real zeros can a cubic function have if it never bends downwards? How many can it have if it has two bends?

   c) How could you use the Intermediate Value Property and the previous parts to determine how many real solutions the equation
   $$x^3 + px^2 + qx + r = 0$$
   has?

d) Without graphing anything, tell how many real zeros these polynomials have:
(i) $f(x) = x^3 + 3x^2 + 3x + 5$

(ii) $g(x) = x^3 - 3x + 1$

(iii) $h(x) = x^3 - 4x + 3$

4. *Extreme deriving—the product rule and beyond.*

a) You know how to take the derivative of $f(x) \cdot g(x)$; you just use the product rule. But how do you take the derivative

$$\frac{d}{dx}[f(x)g(x)h(x)]?$$

What is it?

b) Find the second, third, and fourth derivatives of $f(x)g(x)$. Do you see any pattern in the coefficients you're getting? How do they relate to the coefficients you get when you expand $(x + y)^2$, $(x + y)^3$, and $(x + y)^4$? Can you come up with any reason for the connection?

Scandal in the mathematical world: Have you read the blurb on Newton that comes at the beginning of chapter 3? Check it out to see what Newton did on his summer vacation (well, it wasn't really a *summer* vacation, but it was still while he was on break). Nowadays we give credit to both Sir Isaac Newton and Gottfried Wilhelm von Leibniz for discovering calculus, and we mostly use Leibniz's way of writing things, but there's historically been some controversy about who deserves the credit. For further reading, see http://www.jimloy.com/calc/newtleib.htm .

Selected answers

2. The largest cherry's radius will be 1/2. (See me for more detailed hints on how to prove this mathematically.)

3. a) The graph will have two bends exactly when the derivative equals zero in two different places, i.e. when $3x^2 + 2px + q = 0$ has two real solutions. The discriminant of the quadratic formula tells us that this happens exactly when $4p^2 - 12q > 0$.

   b) one if it never bends downwards; one, two, or three if it has two bends.

   c) We use the answer to part (a) to see if the graph has two bends. If it doesn't, we know there is exactly one real zero of the function. If the function has two bends, then we find the value of the cubic function wherever the graph has a zero derivative (i.e., a bend). If the sign of the function value changes appropriately, we can use the intermediate value property to prove that there are zeros of the cubic to the left of, the right of, and between the two bends.

   d) (i) one real zero; (ii) three real zeros; (iii) three real zeros.

4. a) $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

   b)

$$\frac{d^2}{dx^2} f(x)g(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$$

$$\frac{d^3}{dx^3} f(x)g(x) = f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x)$$

$$\frac{d^4}{dx^4} f(x)g(x) = f^{(4)}(x)g(x) + 4f'''(x)g'(x) + 6f''(x)g''(x) + 4f'(x)g'''(x) + f(x)g^{(4)}(x)$$

   Since we also know

$$(x+y)^2 = x^2 + 2xy + y^2,$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$

   we see the coefficients are the same. Both concepts—that of differentiation of products and of powers of binomials—can be tied to Pascal's triangle.