Merit Worksheet #9, 9/18/06

1. Find the following derivatives:

$$\text{a)}\frac{d}{dx}\left(\frac{3x}{x^3+7x-5}\right) \qquad \text{b)}\frac{d}{dx}\left(\frac{x^3-4x+5}{x^2+9}\right) \qquad \text{c)}\frac{d}{dx}\left(\frac{4x^2-1}{3x(x-1)}\right)$$

2. On the product rule, in taking the derivative of $f(x)g(x)$ you have to add together $f'(x)g(x)$ and $f(x)g'(x)$, and the order in which you add them doesn't matter to your answer. In the quotient rule, you have to combine $f'(x)g(x)$ and $f(x)g'(x)$ somehow as part of your answer. Is the order in which you put these two things important? What happens if you switch the order?

3. a) Find functions $f(x)$ and $g(x)$ so that

$$f(g(x)) = \left(\frac{x+1}{x-1}\right)^7.$$

   b) Find functions $f(x)$, $g(x)$, and $h(x)$ so that

$$f(g(h(x))) = (\sin(\sqrt{x}))^2 + 1.$$

4. Find the following derivatives:

$$\text{a)}\frac{d}{dx}(7-2x^3)^{-4} \qquad \text{b)}\frac{d}{dx}(x^2+3x+4)^3 \qquad \text{c)}\frac{d}{dx}\frac{1}{x^4}\left(1-\frac{1}{x^4}\right)^3$$

5. Suppose $f(x) = (2x+1)^3$. Find $f'(x)$ in the following three ways:

   a) by multiplying out $(2x+1)^3$ and then using the power rule.

   b) by using the product rule on $(2x+1)(2x+1)(2x+1)$.

   c) by using the chain rule.

   Which method did you like the best? Which was was quickest?

6. Suppose $f(x)$ and $g(x)$ are inverse functions, so

$$f(g(x)) = x.$$

   Suppose I know how to take the derivative of $f(x)$, but I don't quite know how to directly find the derivative $g'(x)$. Using the chain rule, find a formula for $g'(x)$.

7. Suppose we're attempting to design a race course with a built-in handicap in the following way: the two runners both start at the origin, with the finish line at $x = 100$. The slower runner, who always runs at 4 units/second, gets to run along the x-axis, while the faster runner has to run at a $30°$ angle to the x-axis. How fast should the faster runner run so that at every instant his x-coordinate is exactly the same as the slower runner's?

8. The top half of a circle with radius R centered at the origin satisfies the equation

$$y = \sqrt{R^2 - x^2}.$$

   a) Find $\frac{dy}{dx}$.

   b) What is the slope of the line segment joining the origin to the point $(a, \sqrt{R^2 - a^2})$?

   c) Using your answer to part (a), what is the slope of the line tangent to the circle at the point $(a, \sqrt{R^2 - a^2})$?

   d) Comparing your answers to parts (b) and (c), what geometric fact have you just proved?

   e) If we wish to draw lines through the point $(2, 2)$ which are tangent to the circle $x^2 + y^2 = 1$, where do these lines hit the circle?

9. Suppose a can of paint has tipped over on a shelf and is spilling paint on the floor at a rate of 1 cubic inch per second. The paint hits the ground and spreads outwards in a perfect circle. If the paint on the ground is 1/2 inch deep everywhere in the circle, at what rate is the radius of the circle growing when the radius is 10 inches? How fast is the area of the circle growing at this same instant?

Quote of the day: "Mathematics is the cheapest science. Unlike physics or chemistry, it does not require any expensive equipment. All one needs for mathematics is a pencil and paper." — George Polya (1887-1985)

Selected answers

1.
a) $$\frac{3(x^3+7x-5)-(3x)(3x^2+7)}{(x^3+7x-5)^2}$$
b) $$\frac{(3x^2-4)(x^2+9)-(x^3-4x+5)(2x)}{(x^2+9)^2}$$
c) $$\frac{8x(3x^2-3x)-(4x^2-1)(6x-3)}{(3x^2-3x)^2}$$

2. Yes; if you switch the order of the subtraction in the quotient rule, you get the wrong sign on your answer.

3. a) $f(x)=x^7$, $g(x)=(x+1)/(x-1)$
   b) $f(x)=x^2+1$, $g(x)=\sin x$, $h(x)=\sqrt{x}$ (other answers are possible).

4. a) $-4(7-2x^3)^{-5}(-6x^2)$
   b) $3(x^2+3x+4)^2(2x+3)$
   c) $-4x^{-5}(1-x^{-4})^3+x^{-4}\cdot 3(1-x^{-4})^2(4x^{-5})$

5. All answers should be equal to $24x^2+24x+6$.

6. $g'(x)=1/f'(g(x))$

7. $8/\sqrt{3}$ units/second

8. a) Using the chain rule and writing the square root as a power, we find
$$\frac{dy}{dx}=\frac{d}{dx}(R^2-x^2)^{1/2}=\frac{1}{2}(R^2-x^2)^{-1/2}(-2x)=\frac{-x}{\sqrt{R^2-x^2}}.$$
Note in the following that the derivative of the stuff under the radical is ***NOT*** $2R-2x$; R^2 is a constant just like 7 or 10 or whatever, since it never changes, so the derivative of R^2 is 0.

   b) To do this we just use the two points we're given, and calculate
$$\text{slope}=\frac{\Delta y}{\Delta x}=\frac{\sqrt{R^2-a^2}-0}{a-0}=\frac{\sqrt{R^2-a^2}}{a}.$$

   c) The slope of the line tangent to the circle is just the derivative, so we plug $x=a$ into our answer to part (a) and get
$$\frac{-a}{\sqrt{R^2-a^2}}.$$

   d) We notice that the two slopes we calculated in parts (b) and (c) are opposite reciprocals of each other; this happens when two lines are perpendicular to each other, so you've just shown that a radius of a circle and a line tangent to the circle at the radius's endpoint are perpendicular.

   e) There's more than one way to do this. Perhaps the simplest way is this: Suppose the tangent line hits the circle at the point $(a,\sqrt{1-a^2})$. (We don't know what a is; in fact, that's what we're trying to find out.) Then the derivative we found in part (a) tells us that the tangent line's slope is
$$\frac{-a}{\sqrt{1-a^2}}.$$

On the other hand, we also know two points on that tangent line—the point $(a, \sqrt{1-a^2})$ and the point $(2, 2)$—so we can calculate the slope using those two points and get

$$\frac{2-\sqrt{1-a^2}}{2-a}.$$

Now both ways should give you the slope, so they should certainly be equal. Setting them equal gives us

$$\frac{-a}{\sqrt{1-a^2}} = \frac{2-\sqrt{1-a^2}}{2-a}.$$

Cross-multiplying gives us

$$a^2 - 2a = 2\sqrt{1-a^2} - (1-a^2).$$

Let's try to get rid of that square root. Moving everything else over to the left-hand side, we find

$$1 - 2a = 2\sqrt{1-a^2};$$

$$1 - 4a + 4a^2 = 4(1-a^2);$$

$$8a^2 - 4a - 3 = 0.$$

To solve this we use the quadratic formula:

$$a = \frac{4 \pm \sqrt{(-4)^2 - 4(8)(-3)}}{16} = \frac{4 \pm 4\sqrt{7}}{16} = \frac{1 \pm \sqrt{7}}{4}.$$

These are the two x-coordinates of the points on the circle where the tangent lines hit. To find the y-coordinates, we can plug these x's into the equation $y = \sqrt{1-x^2}$ and simplify, OR we can notice that the two tangent points should be symmetric about the line $y = x$, so the x-coordinate of one is the y-coordinate of the other, and vice versa. Either way you go about it, you should find that the two tangent points we're after are (equal to, even if they don't look exactly like)

$$\left(\frac{1-\sqrt{7}}{4}, \frac{1+\sqrt{7}}{4}\right) \qquad \text{and} \qquad \left(\frac{1+\sqrt{7}}{4}, \frac{-1+\sqrt{7}}{4}\right).$$

9. Let's examine the paint on the floor. It's a circular layer of paint half an inch thick. That means that the volume of the paint on the ground is

$$\text{Volume} = (\text{Area of the circle})(\text{thickness}),$$

so if $V(t)$ represents the volume of paint on the floor at time t, and r represents the radius of the circle at time t, we have

$$V(t) = \pi r(t)^2 \cdot \frac{1}{2}.$$

Now we are interested in finding dr/dt, i.e. the rate of change of the radius, when the radius is 10. One way to do this is to take the derivative of both sides of the equation above. Then we find

$$\frac{dV}{dt} = \pi \cdot 2r(t) \cdot \frac{dr}{dt} \cdot \frac{1}{2}.$$

Note that because the $r(t)$ was squared in the previous equation, I had to use the chain rule in taking the derivative. Also note that since I don't know a formula for $r(t)$, when I had to take the derivative of it I just put dr/dt. Solving this last equation for dr/dt and plugging in $r = 10$, we see

$$\frac{dr}{dt} = \frac{1}{\pi r(t)} = \frac{1}{10\pi}.$$

Thus the radius is changing at a rate of $1/10\pi$ units per second.

Now I can do something similar to find the rate at which the area is growing, which I illustrate below:

$$A(t) = \pi r(t)^2$$

$$\frac{dA}{dt} = \pi \cdot 2r(t) \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = \pi \cdot 2 \cdot 10 \cdot \frac{1}{10\pi} = 2$$

(Note that in the last line I plugged in what I knew both for r and for dr/dt.) The area is growing at a rate of 2 square inches per second.