Merit Worksheet #12, 9/25/06

In working the first two problems, try using Professor Loeb's "five point path" (aka Loeb's Five Easy Steps) to get a solution. They were in the lecture notes last time. Briefly paraphrased, here they are:

1) Identify the quantity to be maximized or minimized (the problem will tell you this). This is the dependent variable.

2) Write the dependent variable as a function f of one independent variable. You should ask yourself what the connection is between the two—pictures and careful labels are needed here, and you may have to use geometric facts or physical laws to come up with an equation. You may also need to use other equations to eliminate extra variables. Once you've found f, find f's domain, identifying any endpoints.

3) Find all critical points of the function f. Remember, these are (i) the endpoints of the interval of definition, (ii) points where the derivative does not exist, and (iii) points where the derivative exists and is 0.

4) From these critical points, find the point or points where the function takes its maximum or minimum values and evaluate those values.

5) Now answer the question that was asked. Ask yourself—does the answer make sense? Is it possible?

Now on to the problems...

1. A rectangle with sides parallel to the coordinate axes has one vertex at the origin, one on the positive x-axis, one on the positive y-axis, and its fourth vertex in the first quadrant on the line with equation $2x + y = 100$. What is the maximum possible area of such a rectangle?

2. Suppose the post office can accept a cylindrical package for mailing only if the sum of its circumference and its length is at most 100 inches. What is the maximum volume of a cylindrical package that can be mailed? What are the dimensions of such a package?

3. Find the derivatives of the following functions:

   a) $\sin x^2$ b) $\cos^2 x$ c) $\tan(1/x)$ d) $(\sec x)/x$
   e) $\sqrt{\csc x}$ f) $\cot(1 + 2x + 3x^2)$ g) $\sin(\tan(\cos(\sec x)))$

4. In each of the following problems, take the derivatives of the functions in (i) and (ii), and compare their derivatives. Do you see any pattern? Assuming you do, why is there a pattern?

   a) (i) $\sin^2 x + \cos^2 x$ (ii) 1

   b) (i) $1 + \tan^2 x$ (ii) $\sec^2 x$

   c) (i) $1 + \cot^2 x$ (ii) $\csc^2 x$

5. In class you learned that the **arctangent** function, written $\tan^{-1} x$, is the inverse of the tangent function; you stick in a number and the function will tell you what angle that's the tangent of. Now—do you remember the inverse function derivative rule you had on your worksheet a week or two ago? You used the chain rule to prove that if f and g are inverses, so $f(g(x)) = x$, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Using this fact (and perhaps a right triangle), find the derivative of the **arcsine** function, written $\sin^{-1} x$, which is defined to be the inverse of the sine function. What's the domain of your answer? Why does this make sense?

6. Use the logarithm rules to break the following logarithms down into as simple of logarithms as you can:

   a) $\log_{10}\left(\frac{3x^2y^7}{40z^4}\right)$ b) $\ln(17yze^{4x})$ c) $\ln(17x^2/(x^2+1)$

7. Find the derivatives of the following functions:

   a) e^{2x} b) $\ln(x^2+1)$ c) $e^{e^{2x}}$ d) $\ln(17x^2/(x^2+1))$

   (Hint: on part (d) it may be helpful to use part (b) and the previous problem.)

8. Two functions that often show up in engineering or physics settings are the *hyperbolic sine* (sinh) and *hyperbolic cosine* (cosh) functions. They are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

   Find the derivatives of these functions, and explain the similarity these functions share with the regular sine and cosine functions:

   $\frac{d}{dx}\sin x =$ $\frac{d}{dx}\sinh x =$

   $\frac{d}{dx}\cos x =$ $\frac{d}{dx}\cosh x =$

9. A mass suspended by a spring and oscillating about its equilibrium position has position function $x(t)$ given by $x(t) = A\sin(\omega t + \phi_0)$, where A, ω, and ϕ_0 are constants.

   a) Show that the position function $x(t)$ satisfies

$$x''(t) + \omega^2 x(t) = 0.$$

   b) Find the global maximum and the global minimum values of $x(t)$.

10. A simple pendulum consists of a mass m swinging at the end of a (massless) rod or wire of length L. The angular displacement θ (i.e., the angle the rod makes with the vertical) at time t is given by

$$\theta(t) = a\cos(\omega t + \phi),$$

   where a, ω, and ϕ are constants. Show that θ satisfies the equation

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0.$$

A sad story. Our friend π walks into a party and sees all kinds of functions participating in the new craze—they're differentiating themselves. There's $\ln x$, who differentiates himself and becomes $1/x$; x^2 becomes $2x$. Old e^x sits in a corner, moaning, "It's all the same; it's all the same." So π differentiates himself and promptly becomes 0. The moral: just one time *can* kill you. Differentiate responsibly.

Selected answers

1. 1250 units^2

2. Maximum volume $1000000/(27\pi)$ cubic inches, attained when the radius is $100/(3\pi)$ inches and the length is $100/3$ inches.

3. a) $\cos(x^2) \cdot 2x$
   b) $2\cos x \cdot -\sin x$
   c) $\sec^2(1/x) \cdot -\frac{1}{x^2}$
   d) $(x \sec x \tan x - \sec x)/x^2$
   e) $\frac{1}{2}(\csc x)^{-1/2} \cdot -\csc x \cot x$
   f) $-\csc^2(1 + 2x + 3x^2) \cdot (2 + 6x)$
   g) $\cos(\tan(\cos(\sec x))) \sec^2(\cos(\sec x)) \cdot -\sin(\sec x) \cdot \sec x \tan x$

4. a) $2\sin x \cos x + 2\cos x \cdot -\sin x = 0$
   b) $0 + 2\tan x \sec^2 x = 2\sec x \sec \tan x$
   c) $0 + 2\cot x \cdot -\csc^2 x = 2\csc x \cdot -\csc x \cot x$

   in each case the derivatives are equal because the two quantities are identically equal.

5. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$; domain is $(-1, 1)$, and this makes sense because the sine only takes on values between -1 and 1.

6. a) $\log_{10} 3 + 2\log_{10} x + 7\log_{10} y - \log_{10} 40 - 4\log_{10} z$
   b) $\ln 17 + \ln y + \ln z + 4x$
   c) $\ln 17 + 2\ln x - \ln(x^2 + 1)$

7. a) $e^{2x} \cdot 2$
   b) $\frac{1}{x^2+1} \cdot 2x$
   c) $e^{e^{2x}} \cdot e^{2x} \cdot 2$
   d) $2/x - \frac{1}{x^2+1} \cdot 2x$

8. $\frac{d}{dx} \sinh x = \cosh x$, $\frac{d}{dx} \cosh x = \sinh x$

9. a) $-A\omega^2 \sin(\omega t + \phi_0) + \omega^2 A \sin(\omega t + \phi_0) = 0.$
   b) Global maximum of A; global minimum of $-A$.

10. $-a^2 \cos(\omega t + \phi) \cdot \omega^2 + \omega^2 a \cos(\omega t + \phi) = 0.$