Merit Worksheet #17, 10/9/06

Newton's Method

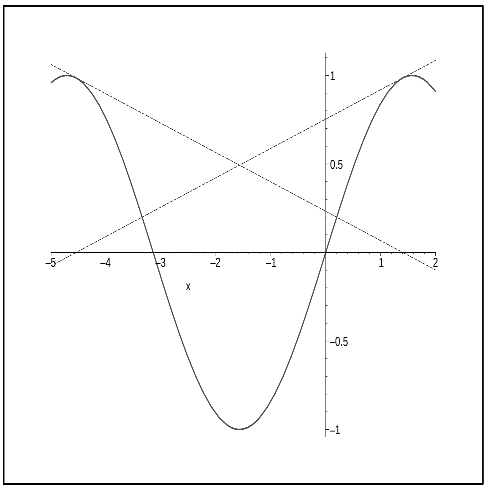
1. Given a differentiable function $y = f(x)$, you want to find a value x such that $f(x) = 0$. You make a guess x_1 such that the derivative $f'(x_1) \neq 0$.

   a) Write the formula for the tangent line ℓ at the point $(x_1, f(x_1))$.

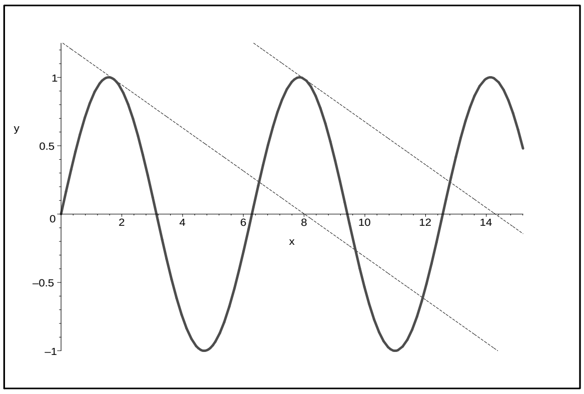
   b) Your next guess, x_2, is the x-coordinate of the point where that tangent line ℓ intersects what line?

   c) Using your answers to (a) and (b), find a formula for x_2.

2. Suppose x_1 and x_2 are the x-coordinates shown below (the function graphed is $f(x) = \sin x$. Why won't Newton's method work with this choice of x_1?



3. (**Bonus question**—come back to this one after the rest of the worksheet is done) Suppose your friend asks you how to use Newton's method to find solutions to $\sin x = 0$, but you, with your perverse sense of humor, decide to give him an initial guess x_1 chosen carefully so that x_2 is exactly 2π units later (so x_3 will be 2π units after that, and the sequence of x_n's will never settle down to a solution). What should x_1 be?



Differentials and linear approximation

4. Compute the following differentials:

   a) $d(\sin x)$ b) $d\left(e^{x^2} + \frac{3}{x}\right)$

5. For each of the following functions, write dy in terms of x and dx.

   a) $y = \cos\sqrt{2x}$ b) $d(x \tan x)$

6. Use linear approximation to estimate the following values (remember to convert angles to radian measure).

   a) $\sqrt{65}$ b) $\sin 46°$

7. Use differentials to estimate the change in the area of a square, if its edge length is decreased from 10 in. to 9.8 in.

8. (**Bonus question**—come back to this one after the rest of the worksheet is done) We are trying to determine the area of a circle by measuring the diameter. How accurately must we measure the diameter (i.e., with what percentage of error) if our estimate for the area is to be within 1% of the true value?

The Mean Value Theorem

9. Explain why the function $f(x) = (x-1)^{2/3}$ satisfies the **hypothesis** of the Mean Value Theorem on the interval $[1, 2]$, and find the place(s) c where

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}.$$

10. If f is differentiable on $[3, 7]$, $f(3) = -1$, and $f(7) = 11$, is it possible that $f'(x) \geq 3$ for all x in $[3, 7]$? Is it possible that $f'(x) \leq 2$ for all $x \in [3, 7]$? Why or why not?

11. (**Bonus question**—come back to this one after the rest of the worksheet is done) Prove that for all real x and y,

$$|\cos x - \cos y| \leq |x - y|.$$

Comment

Next time's worksheet will cover higher derivatives (e.g., finding the second derivative) and the First Derivative Test. **As part of your preparation points for next time,** you will need to come with a short explanation (written in complete sentences) of how you can use the first derivative to decide if a critical point of a function $f(x)$ is a local maximum or local minimum. This must be turned in at the beginning of class—no late responses will be accepted.

Quote of the day: "The Mean Value Theorem is the midwife of calculus — not very important or glamorous by itself, but often helping to delivery other theorems that are of major significance." — E. Purcell and D. Varberg

Selected answers

1. a) $y - f(x_1) = f'(x_1)(x - x_1)$

   b) the x-axis

   c)
$$x_2 = x_1 + \frac{0 - f(x_1)}{f'(x_1)} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

2. Because x_3 is exactly back where x_1 was; the process just keeps jumping back and forth between x_1 and x_2, never getting any closer to where $\sin x = 0$.

3. There are a couple of answers here. One is $x_1 = \arctan(-2\pi)$; the one illustrated is $x_1 = \pi + \arctan(-2\pi)$.

4. a) $\cos x \, dx$

   b) $\left(e^{x^2} \cdot 2x - \frac{3}{x^2}\right) dx$

5. a) $dy = -\sin(\sqrt{2x}) \cdot \frac{1}{2}(2x)^{-1/2} \cdot 2 \, dx$

   b) $dy = (\tan x + x \sec^2 x) \, dx$

6. a) $8\frac{1}{16} = 8.0625$

   b) $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\pi}{180} \approx 0.7194$

7. $dA = 2x \, dx = 2(10)(-0.2) = -4 \text{ in}^2$

8. Within 0.5% of the true value.

9. The function is continuous on the closed interval $[1, 2]$ and differentiable on the open interval $(1, 2)$; $c = 35/27$.

10. Since
$$\frac{f(7) - f(3)}{7 - 3} = \frac{11 - (-1)}{7 - 3} = 3,$$
the Mean Value Theorem says there's a point between 3 and 7 where the derivative exactly equals 3. Therefore, it's *not* possible that $f'(x) \leq 2$ for all x between 3 and 7. It *is* possible that $f'(x) \geq 3$; in fact, the straight line passing through $(3, -1)$ and $(7, 11)$ is a continuous and differentiable function and has slope 3 everywhere.

11. By the mean value theorem, there's a place c between x and y where
$$\frac{\cos x - \cos y}{x - y} = -\sin c.$$
Since $-1 \leq -\sin c \leq 1$, we see that
$$\left|\frac{\cos x - \cos y}{x - y}\right| \leq 1,$$
and therefore
$$|\cos x - \cos y| \leq |x - y|.$$