Merit Worksheet #18, 10/11/06

Second Derivatives

1. Calculate the second derivative of the following functions:

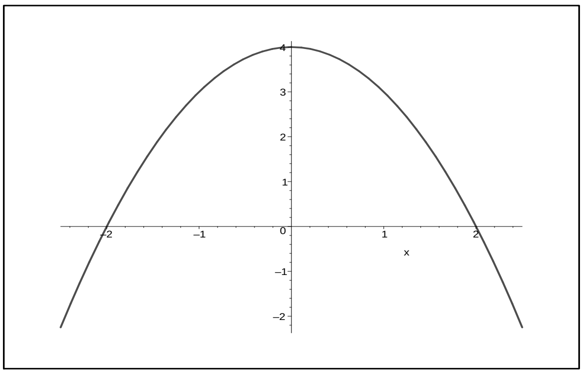
$$\text{a)} \quad f(t) = t^2 \ln t \qquad\qquad \text{b)} \quad g(x) = \frac{\sin x}{x}.$$

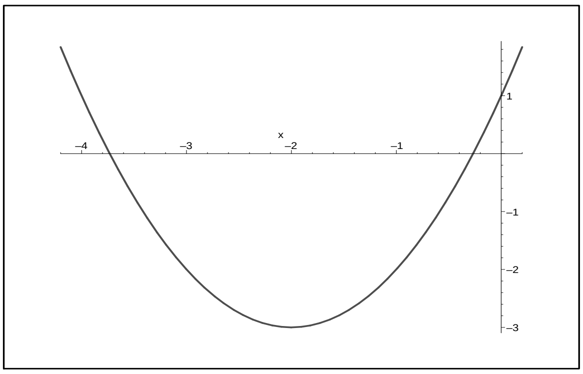
2. Find $\frac{d^2y}{dx^2}$ using implicit differentiation on the equation $x^2 + y^3 + y = 1$.

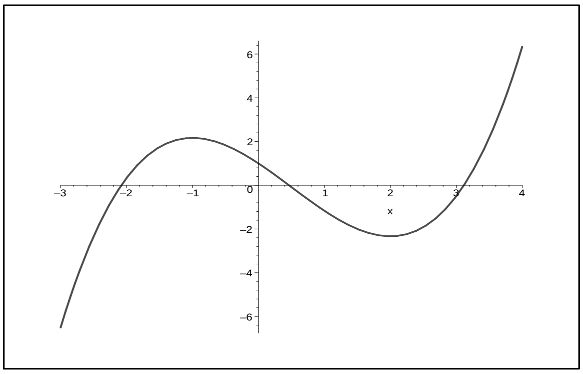
Increasing/Decreasing Functions

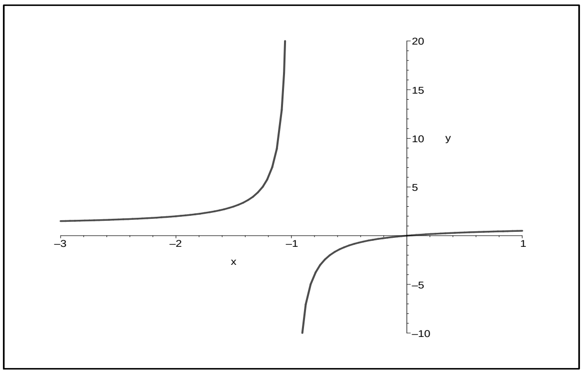
3. Use the first derivative to find where the following functions are increasing and decreasing. Then, **and only then**, use this information to match the following functions with their graphs:

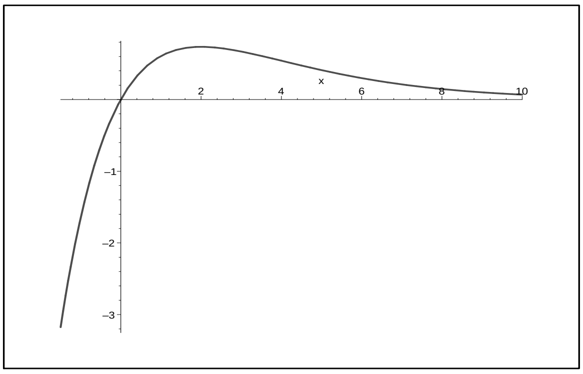
a) $f(x) = x/(x+1)$ b) $g(x) = x^2 + 4x + 1$ c) $h(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 1$
d) $j(x) = 4 - x^2$ e) $k(x) = xe^{-x/2}$





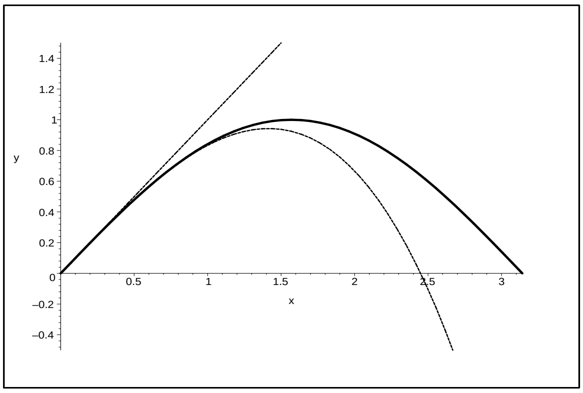






4. a) Show, using the first derivative, that $\sin x > x - \frac{1}{6}x^3$ for all positive x.

   b) In your book it's proved that $\sin x < x$ for all positive x. This means that the graph of the sine function is sandwiched (squozed?) between $y = x$ and $y = x - \frac{1}{6}x^3$, like so:



   Use this information (and a calculator) to calculate the sine of a 5° angle accurate to three decimal places, and check your answer with the true value.

The First Derivative Test

5. Apply the first derivative test to classify each of the critical points of the following functions as a local maximum or minimum, a global maximum or minimum, or not an extremum.

   a) $f(x) = x^3 - 3x^2 + 3x + 5$ b) $f(x) = 10 + 60x + 9x^2 - 2x^3$ c) $f(x) = (1 - \ln x)/x$

6. What does the first derivative test tell you about maxima and minima of a function, that just finding the critical points and plugging them in didn't? In other words, why are we looking at maximization/minimization again, when we already covered it in the last chapter?

7. Each page of a book will contain 30 in.^{2} of print, and each page must have 2-in. margins at the top and bottom and a 1-in. margin at each side. What is the minimum possible area of such a page?

Quote of the day: "In the company of friends, writers can discuss their books, economists the state of the economy, lawyers their latest cases, and businessmen their latest acquisitions, but mathematicians cannot discuss their mathematics at all. And the more profound their work, the less understandable it is." — Alfred Adler

Selected answers

1. a) $f''(t) = 3 + 2\ln t$

   b)
$$g''(x) = \frac{(\cos x - x\sin x - \cos x)x^2 - 2x(x\cos x - \sin x)}{x^4}$$

2.
$$\frac{d^2y}{dx^2} = \frac{-2 - 6y\left(\frac{-2x}{3y^2+1}\right)^2}{3y^2+1}$$

3. a) Increasing on $(-\infty, -1)$ and on $(-1, \infty)$; graph is the second one in the second row.

   b) Decreasing on $(-\infty, -2)$, increasing on $(-2, \infty)$; graph is the second one in the first row.

   c) Increasing on $(-\infty, -1)$, decreasing on $(-1, 2)$, increasing on $(2, \infty)$; graph is the first one in the second row.

   d) Increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$; graph is the first one in the first row.

   e) Increasing on $(-\infty, 2)$, decreasing on $(2, \infty)$; graph is the one in the third row.

4. a) Let $f(x) = \sin x - (x - x^3/6)$. Then $f'(x) = \cos x - 1 + x^2/2$. Setting the derivative equal to zero, we see that $x = 0$ is a critical point. It turns out that that's the *only* critical point, but that's a lot of work to show; sorry about that—let's just take it on faith that $x = 0$ is the only critical point. Plugging in an x to the right of $x = 0$ (say $x = 100$), you see that $f'(100)$ is much bigger than zero, so the slopes of $f(x)$ are all positive when $x > 0$. This means that $f(x)$ is always increasing when $x > 0$. Now at 0, $f(0) = \sin 0 - (0 - 0^3/6) = 0$, so since $f(x)$ is increasing, we see that $f(x) > 0$ whenever $x > 0$. Then
$$f(x) = \sin x - (x - x^3/6) > 0,$$
so
$$\sin x > x - x^3/6,$$
and we're done. Neat, huh?

   b) Since $5°$ is the same as $5(\pi/180) = \pi/36$ radians, we find
$$\frac{\pi}{36} - \frac{1}{6}\left(\frac{\pi}{36}\right)^3 < \sin\frac{\pi}{36} < \frac{\pi}{36}.$$
Using a calculator,
$$0.08716 < \sin\frac{\pi}{36} < .08727$$
We can conclude that, correct to three decimal places,
$$\sin 5° = 0.087$$

5. a) The only critical point is $x = 1$; it's not an extremum.

   b) The critical points are $x = -2$ and $x = 5$; the first is a local minimum, the second is a local maximum.

   c) Note that only positive numbers are in the domain of the function. The only critical point in the domain is $x = e^2$; this is a global minimum.

6. What we did in the last chapter allowed us to find the *global* maximum and *global* minimum of functions on closed intervals (there were a few shortcuts we used to find out a bit more about functions, but basically that's it for what we learned in the last chapter). With the first derivative test you can also decide if a point is a *local* (but not necessarily global) max/min of a function, and you can do it for functions that aren't necessarily on closed intervals—you can handle open intervals, or the whole real line, or whatever. In short, we can do a whole lot more than we could before.
7. This will appear on next time's worksheet.