Merit Worksheet #19, 10/13/06

The First Derivative Test

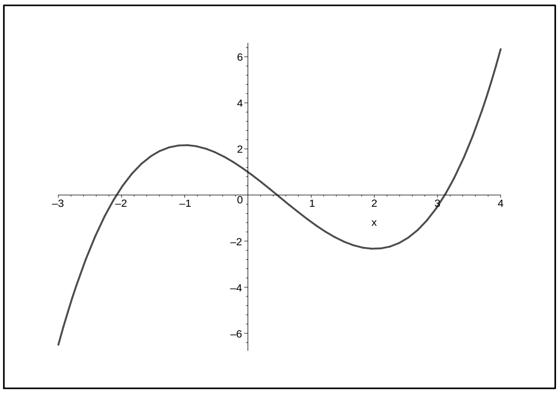
1. Each page of a book will contain 30 in.^{2} of print, and each page must have 2-in. margins at the top and bottom and a 1-in. margin at each side. What is the minimum possible area of such a page?

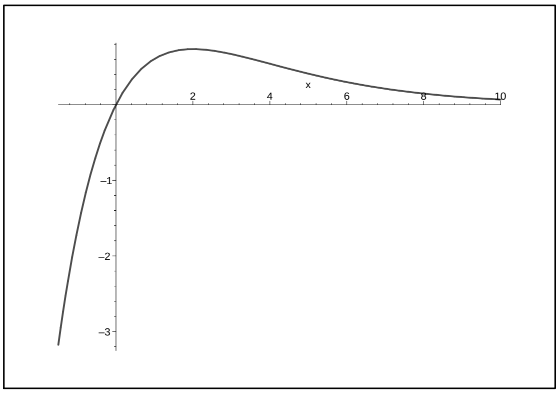
The Second Derivative

2. Let $f(x)$ be some function. Match the phrases below to the mathematical expressions that mean the same thing.

f's slope is increasing around $x = a$	$f(x) = c$
$f(x)$ is decreasing around $x = a$	$f''(a) = 0$
$x = a$ is the location of a critical point	$f'(a) > 0$
The graph of $f(x)$ is concave down around $x = a$	$f'(a) < 0$
$f(x)$ is constant	$f''(a) > 0$
$x = a$ may be the location of an inflection point	$f'(a) = 0$
$f(x)$ is getting bigger near $x = a$	$f''(a) < 0$

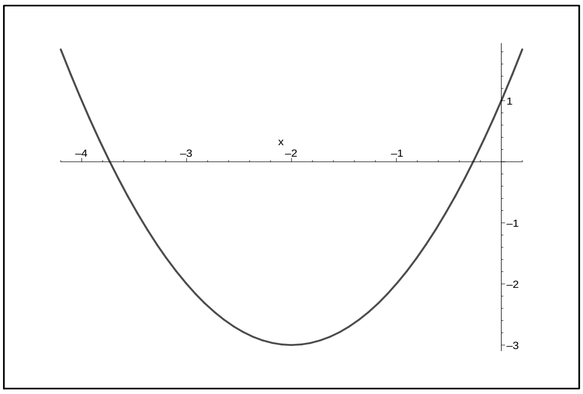
3. a) By looking carefully at the graphs below, mark the approximate location of any inflection points:





b) The graphs above are of the functions $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 1$ and $g(x) = xe^{-x/2}$, respectively. Find the exact location of the inflection points by using the second derivative.

4. a) Below is a sketch of $f'(x)$, the ***derivative*** of $f(x)$. Using this information, state where the local extrema and inflection points are **in the original function $f(x)$.**



b) If you are told that $f(0) = 0$, use the information from part (a) to give a rough sketch of $f(x)$.

5. (Bonus Question—come back to this one when you're done with the rest) Prove that if a cubic function $f(x) = ax^3 + bx^2 + cx + d$ has two separate critical points, then $f(x)$'s inflection point is the midpoint of the line segment joining f's local maximum and local minimum points.

Asymptotes

6. How can you tell where the vertical asymptotes (if any) are for a function? The horizontal asymptotes?

7. Find all asymptotes of the following functions:

a) $f(x) = \dfrac{\sin x}{x}$ b) $g(x) = \dfrac{3x^3 - 3}{4x^3 - 12x^2 + 8x}$

Curve sketching

8. Consider the function $f(x) = \frac{x}{1-x^2}$.

a) Find all x and/or y-intercepts of the function.
b) Identify any asymptotes of $f(x)$.
c) Specify the intervals on which $f(x)$ increases, and those on which it decreases.
d) Specify where $f(x)$ is concave up and concave down, and all points of inflection.
e) Sketch a pretty accurate graph of $f(x)$, using the information from parts (a)-(d).

A little humor.

• Q: How do you make one burn?
A: Differentiate a log fire!

• Mathematicians never die - they only lose some of their functions.

• *New York (CNN).* At John F. Kennedy International Airport today, a Caucasian male (later discovered to be a high school mathematics teacher) was arrested trying to board a flight while in possession of a compass, a protractor and a graphical calculator. According to law enforcement officials, he is believed to have ties to the Al-Gebra network. He will be charged with carrying weapons of math instruction.

Selected answers

1. $38 + 8\sqrt{15}$

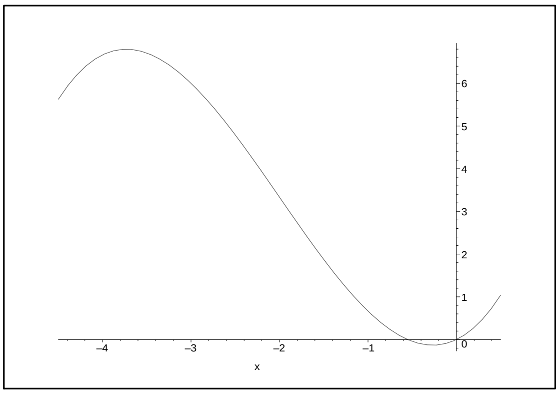
2. Matched up, the answers are as follows:

f's slope is increasing around $x = a$	$f''(a) > 0$
$f(x)$ is decreasing around $x = a$	$f'(a) < 0$
$x = a$ is the location of a critical point	$f'(a) = 0$
The graph of $f(x)$ is concave down around $x = a$	$f''(a) < 0$
$f(x)$ is constant	$f(x) = c$
$x = a$ may be the location of an inflection point	$f''(a) = 0$
$f(x)$ is getting bigger near $x = a$	$f'(a) > 0$

3. b) The inflection points are at $(1/2, 0)$ and at $(4, 4e^{-2})$, respectively.

4. a) The critical points of $f(x)$ are where $f'(x) = 0$, i.e. at the x-intercepts of the graph. These are at approximately $x = -3.7$ and $x = -0.3$. Since $f'(x) > 0$ when $x < -3.7$ and when $x > -0.3$, the original function $f(x)$ is increasing on these intervals. Since $f'(x) < 0$ when $-3.7 < x < -0.3$, the original function $f(x)$ is decreasing on this interval. Thus there is a local maximum at about $x = -3.7$, and a local minimum at about $x = -0.3$. At $x = 2$ the slope on the graph shown is 0, so $0 = (f'(x))'|_{x=-2} = f''(-2)$, so $f(x)$ may have an inflection point at $x = -2$. It does, in fact, since the second derivative of $f(x)$ is the same as the derivative of $f'(x)$, which is negative when $x < -2$ and positive when $x > 2$.

   b) Hopefully your graph of $f(x)$ looks something like this:



5. The two critical points (let's call them x_1 and x_2) of $f(x)$ have x-coordinates

$$x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = -\frac{b}{3a} \pm \frac{\sqrt{b^2 - 3ac}}{3a};$$

the inflection point of $f(x)$ has x-coordinate $x_0 = -b/3a$, so you can see that x_1 and x_2 are evenly spaced horizontally on either side of x_0. Plugging in each of these three x's into $f(x)$, it's messy, but you can see that the average of $f(x_1)$ and $f(x_2)$ is the same as

$$f(x_0) = \frac{2b^3 - 9abc + 27da^2}{a^2},$$

so the y-coordinate of the inflection point is exactly halfway between the y-coordinates of the two critical points. Since the inflection point's x- and y-coordinates are smack in the middle of the x- and y-coordinates of the critical points, the inflection point is the midpoint of the line segment joining the two other points.

6. See your text, pages 272-273 for the vertical asymptote stuff, and pages 274-276 for the horizontal asymptote. If you find the horizontal asymptote stuff confusing, just pay attention to the first sentence of the "Horizontal Asymptotes" section on page 274, and look over the examples.

7. a) The only possible place this function could have a vertical asymptote would be at $x = 0$; however, it does not have an asymptote at $x = 0$, because

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

instead of $\pm\infty$. The function has a horizontal asymptote of $y = 0$, since

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

and

$$\lim_{x \to -\infty} \frac{\sin x}{x} = 0.$$

b) We have a vertical asymptote whenever the denominator equals zero but the numerator does not; the denominator factors as

$$4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2),$$

so the function is undefined at $x = 0$, $x = 1$, and $x = 2$. However, $x = 1$ also makes the numerator equal to zero, so the only vertical asymptotes of this function are at $x = 0$ and $x = 2$ (there's just a hole in the graph at $x = 1$). To find the horizontal asymptote, we see that

$$\lim_{x \to \infty} \frac{3x^3 - 3}{4x^3 - 12x^2 + 8x} = \lim_{x \to \infty} \frac{3 - \frac{3}{x^3}}{4 - \frac{12}{x} + \frac{8}{x^2}} = \frac{3}{4},$$

so the function has a horizontal asymptote of $y = 3/4$.

8. This question will appear on a later worksheet.