Merit Worksheet #20, 10/16/06

Curve sketching

1. Consider the function $f(x) = \frac{x}{1-x^2}$.

   a) Find all x and/or y-intercepts of the function.

   b) Identify any asymptotes of $f(x)$.

   c) Specify the intervals on which $f(x)$ increases, and those on which it decreases.

   d) Specify where $f(x)$ is concave up and concave down, and all points of inflection.

   e) Sketch a pretty accurate graph of $f(x)$, using the information from parts (a)-(d).

l'Hôpital's Rule

2. Find the following limits, using l'Hôpital's rule where appropriate:

   a) $\displaystyle\lim_{x\to\infty} \frac{3x-4}{2x-5}$ b) $\displaystyle\lim_{x\to 0} \frac{e^{3x}-1}{x}$ c) $\displaystyle\lim_{x\to 0^+} \frac{1-\cos\sqrt{x}}{x}$

3. Find the following limits, using l'Hôpital's rule where appropriate:

   a) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{4x^2-x}}$ b) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x+x^2}$ c) $\displaystyle\lim_{x\to 2\pi} \frac{\sin x}{x-\pi}$

4. Find the following limits, using l'Hôpital's rule where appropriate:

   a) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$ b) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x+1} - \sqrt{x}\right)$

5. Recall that the *local linearization* of a function $f(x)$ near $x = a$ is given by

$$f(x) \approx f(a) + f'(a)(x-a).$$

   a) If $f(x)$ and $g(x)$ are both differentiable functions with $f(a) = 0$ and $g(a) = 0$, what are the local linearizations of $f(x)$ and $g(x)$ near $x = a$?

   b) Using your answer to part (a), what would you expect the fraction $f(x)/g(x)$ to be approximately equal to near $x = a$?

   c) How does your answer to part (b) agree with l'Hôpital's rule?

Antiderivatives

6. Find the following antiderivatives:

   a) $\displaystyle\int \frac{1}{x}\,dx$ b) $\displaystyle\int \frac{1}{3}x^3\,dx$ c) $\displaystyle\int \cos x\,dx$ d) $\displaystyle\int e^x\,dx$ e) $\displaystyle\int \frac{1}{1+x^2}\,dx$

7. Find the following antiderivatives:

   a) $\displaystyle\int \left(\frac{3}{x^3} + 2x^{3/2} - 1\right)dx$ b) $\displaystyle\int \frac{1}{(x-10)^7}\,dx$ c) $\displaystyle\int (3\cos\pi t + \cos 3\pi t)\,dt$

8. Remember how complicated it was to take the derivative of a product, quotient, or function inside of a function? We had to develop the product, quotient, and chain rules. It will be similarly complicated to find the antiderivative of products and functions inside of functions. (This is what we'll learn about in the rest of this course and in the beginning of Calc II).

   Take the antiderivative of each of the following expressions, and compare your answer to the rule found on the stated page. How are your answer and the book's rule the same?

   a) $\frac{d}{dx}u(x)v(x) = u(x)v'(x) + u'(x)v(x)$; the blue box at the bottom of page 496 in your text.
   b) $\frac{d}{dx}h(g(x)) = h'(g(x))g'(x)$; the top 4 lines of page 361 in your text.

Extra practice

You've noticed, probably, that our merit problems these days more closely resemble the homework than they did at the beginning of the semester. While I plan to keep this up, I'm still going to include some problems to make you think a bit more than your homework does. With that said, hopefully you realize that you most likely won't get through enough problems in our workshops to make you an expert on what we cover. For that reason I hope to include a list of suggested exercises at the end of most worksheets. If you're reviewing for the test sometime later and know you're weak in a certain area, or you're done with your homework and want a little more practice in a particular type of problem, you might try the following...

Curve sketching
• (Re)read the "Curve-Sketching Strategy" steps on page 276, and example 8 on pages 276-277.
• There are plenty of problems in sections 4.5 through 4.7 that resemble your homework and are good practice. Pick any of them and try them; my suggestions, to get you started, are Section 4.5, problems 7, 11, 21, 25, and 43; Section 4.6, problems 63, 67, 71, 77, and 81; Section 4.7, problems 23, 25, 39, 41, and 47.

l'Hôpital's rule
• (Re)read Section 4.8 through the bottom of page 284, and examples 1, 2, 3, and 6.
• Work any of problems 1–48 in section 4.8. Specifically, try problems, 1, 5, 9, 15, and 19. Then work problems 13, 15, and 17 in section 4.9.

Antiderivatives
• (Re)read page 303 in your text, and Example 1 on the next page, and skim the rest of the material on pages 304–307.
• Work any of the problems 1–30 in section 5.2. Specifically try problems 1, 3, 13, 23, and 27.

A little scandal in math history. l'Hôpital's rule isn't really l'Hôpital's. According to the article at `http://en.wikipedia.org/wiki/L'Hopital's_rule` ,

"The rule is believed to be the work of Johann Bernoulli. L'Hôpital, a nobleman, paid Bernoulli a retainer of 300 pounds per year to keep him updated on developments in calculus and to solve problems he had. Among these problems was that of limits of indeterminate forms. When l'Hôpital published his book [which, let me jump in here and point out, was the first textbook on calculus ever written], he gave due credit to Bernoulli and, not wishing to take credit for any of the mathematics in the book, he published the work anonymously. Bernoulli, who was known for being extremely jealous, claimed to be the author of the entire work, and until recently, it was believed to be so. Nevertheless, the rule was named for l'Hôpital, who never claimed to have invented it in the first place."

So, the next time somebody asks you which famous mathematician you'd most like to have lunch with, go for a Newton or a Bernoulli. Don't even *think* about answering, "l'Hôpital;" you'll only be setting yourself up for disappointment.

Selected answers

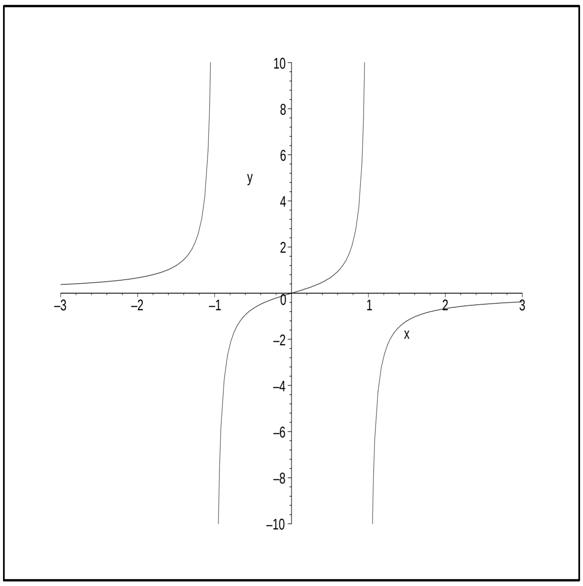
1. a) Only x-intercept, *and* the only y-intercept, is $(0, 0)$

   b) Vertical asymptotes of $x = -1$ and $x = 1$; horizontal asymptote of $y = 0$.

   c) Increases on each of $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$

   d) Concave up on $(-\infty, -1)$ and $(0, 1)$; concave down on $(-1, 0)$ and $(1, \infty)$; point of inflection at $(0, 0)$.

   e) The graph looks like this:



2. a) 3/2

   b) 3

   c) 1/2

3. a) 1/2

   b) 1

   c) 0 (note that you CANNOT use l'Hôpital's rule on this one!)

4. a) 1/2

   b) 0

5. a)

$$f(x) \approx f'(a)(x - a)$$
$$g(x) \approx g'(a)(x - a)$$

   b) Near $x = a$,

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x - a)}{g'(a)(x - a)} = \frac{f'(a)}{g'(a)}$$

   c) L'Hôpital's rule says that $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, so the limit is equal to the ratio of the derivatives, just as part (b) suggests.

6. a) $\ln x + C$

b) $\frac{1}{12}x^4 + C$

c) $\sin x + C$

d) $e^x + C$

e) $\tan^{-1} x + C$

7. a) $-\frac{3}{2x^2} + \frac{4}{5}x^{5/2} - x + C$

b) $-\frac{1}{6(x-10)^6} + C$

c) $\frac{3}{\pi}\sin \pi t + \frac{1}{3\pi}\sin 3\pi t + C$